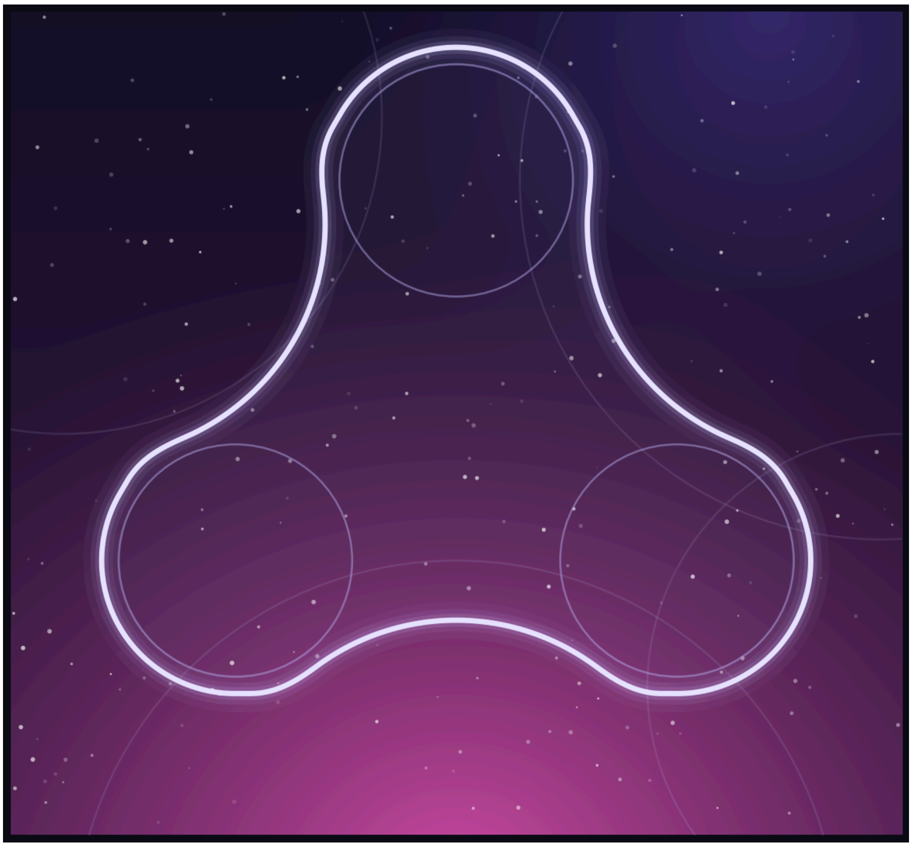
Smoothly Enrobing Circles

Implicit Curves, Smooth Booleans, and the Geometry of the *Destiny 2: Lightfall* Title-Screen Tracery



A single smooth, glowing contour wrapping three circles — the object this report constructs, rendered in the title-screen aesthetic.

Abstract

We want one smooth line that wraps a small set of circles the way the *Lightfall* menu background does: hugging each circle, then flowing across the gaps. We treat the line as the *level set* of a scalar field and build that field in four stages, each fixing a defect of the last: additive metaballs, a smooth-minimum of signed distance fields, the convex hull of the disks (a taut belt), and finally the *morphological closing* that bends each bridge inward around an "invisible circle." A smooth Boolean operator removes the curvature cusps at the junctions, and a marching-squares plus quadratic-spline pipeline turns the field into a clean curve. All formulae used are derived below.

1 Goal

Given N circles with centres c_i and radii R_i, produce a single closed curve that

1. rides along each circle (it "enrobes" them),
2. joins neighbouring circles with a smooth bridge, and
3. is *tangent-continuous* (G^1) everywhere, so it reads as flowing rather than drawn.

The reference is the *Lightfall* title screen, whose tracery shows circles linked by large, gentle, tangent arcs. The whole report restricts to equal radii $R_i = R$, which is the case that arises in the motivating artwork and which makes several formulae collapse to closed form.

2 Implicit curves and level sets

Rather than parameterising the boundary, we describe it implicitly. For a scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ and a threshold c, the curve is

$$\Gamma \;=\; \{\, p \in \mathbb{R}^2 : f(p) = c \,\}. \tag{1}$$

The key fact we lean on throughout is the **implicit function theorem**: if f is C^k and $\nabla f \neq 0$ along Γ, then Γ is locally a C^k curve. *The smoothness of the outline is inherited from the smoothness of the field.* This is why every "how do I make it smoother" question below reduces to "how do I make the field smoother." The signed curvature of a level set is itself an expression in the field's derivatives,

$$\kappa \;=\; \frac{f_x^2 f_{yy} - 2 f_x f_y f_{xy} + f_y^2 f_{xx}}{\left(f_x^2 + f_y^2\right)^{3/2}}, \tag{2}$$

so a discontinuity in the field's second derivatives shows up as a discontinuity in κ — the origin of the cusps we fix in §7.

3 First attempt: additive metaballs

Give each circle an influence that decays with distance and sum them:

$$f(p) \;=\; \sum_{i=1}^{N} \frac{R_i^2}{\|p - c_i\|^2}, \qquad \Gamma : \; f(p) = T. \tag{3}$$

For a single isolated circle, $f = R^2/d^2 = T$ gives $d = R/\sqrt{T}$, so $T = 1$ recovers the circle exactly and $T < 1$ produces a uniform halo at radius $R/\sqrt{T}$. When two circles are near, their fields *add* in the gap, pushing the contour outward and forming a bridge (Fig. 2a). Two defects appear:

- **The coupling is global.** The $1/d^2$ falloff is long-ranged, so distant circles deform one another — the connection feels "too strong."
- **The merge passes through a saddle.** Where the two blobs fuse, ∇f is small; by (2) the curvature spikes and the neck pinches to a near-cusp.

We need a field whose blend is *local* and whose merge is gentle.

4 Smooth-minimum of signed distance fields

Replace "sum of influences" with "smooth union of distances." Each circle gets a signed distance function (SDF), negative inside,

$$d_i(p) \;=\; \|p - c_i\| - R_i, \tag{4}$$

and we combine them with the polynomial *smooth minimum* (the quadratic, C^1 form):

$$\boxed{\operatorname{smin}_k(a, b) \;=\; \min(a, b) \;-\; \tfrac{1}{4}\, k\, h^2, \qquad h = \max\!\Big(1 - \tfrac{|a-b|}{k},\, 0\Big)} \tag{5}$$

folded over the circles, $D(p) = \operatorname{smin}_k\big(d_1(p), \ldots, d_N(p)\big)$, with outline $D(p) = o$. The offset o is a Minkowski dilation: $o = 0$ traces the surfaces, $o > 0$ a coat of width o. Two properties matter:

- **Locality.** The correction term vanishes when $|a - b| \geq k$, so $\operatorname{smin}_k = \min$ away from junctions; only nearby circles blend. This cures the global coupling.
- **Gentle merge.** The blend introduces a fillet of radius $\sim k$, so the neck's curvature is bounded, $\kappa_{\max} \approx 1/k$ — no saddle cusp (Fig. 2b).

We will reuse its dual, the *smooth maximum*,

$$\operatorname{smax}_k(a, b) \;=\; -\, \operatorname{smin}_k(-a, -b) \;=\; \max(a, b) + \tfrac{1}{4} k\, h^2, \tag{6}$$

in §7. This already matches the screen well; setting $k \approx 2R$ makes the fillet several times the circle radius, which is the gentle sweep the artwork uses.

5 The taut belt: convex hull of the disks

The smin neck pinches *inward*. If instead we want the line to ride the *outside* of the circles and cross each gap on a taut tangent — a belt around pulleys — the right object is the convex hull of the disks. For equal radii the external tangents are parallel to the centre line (offset by R), so the belt is built from straight tangent segments and circular arcs; on two equal circles each arc is exactly a **semicircle** (the outer 180°). Compactly, the m-coat outline is the level set of the distance to the hull of the centres, equivalently a Minkowski sum,

$$\Gamma \;=\; \big\{\, p : \mathrm{dist}\big(p,\ \mathrm{conv}\{c_i\}\big) = R + m \,\big\} \;=\; \mathrm{conv}\{c_i\} \oplus \mathcal{D}_{R+m}. \tag{7}$$

Its curvature is constant $1/(R+m)$ on the arcs and 0 on the straights, so it is G^1 (Fig. 2c). For two circles this is a stadium; for three, a rounded triangle.

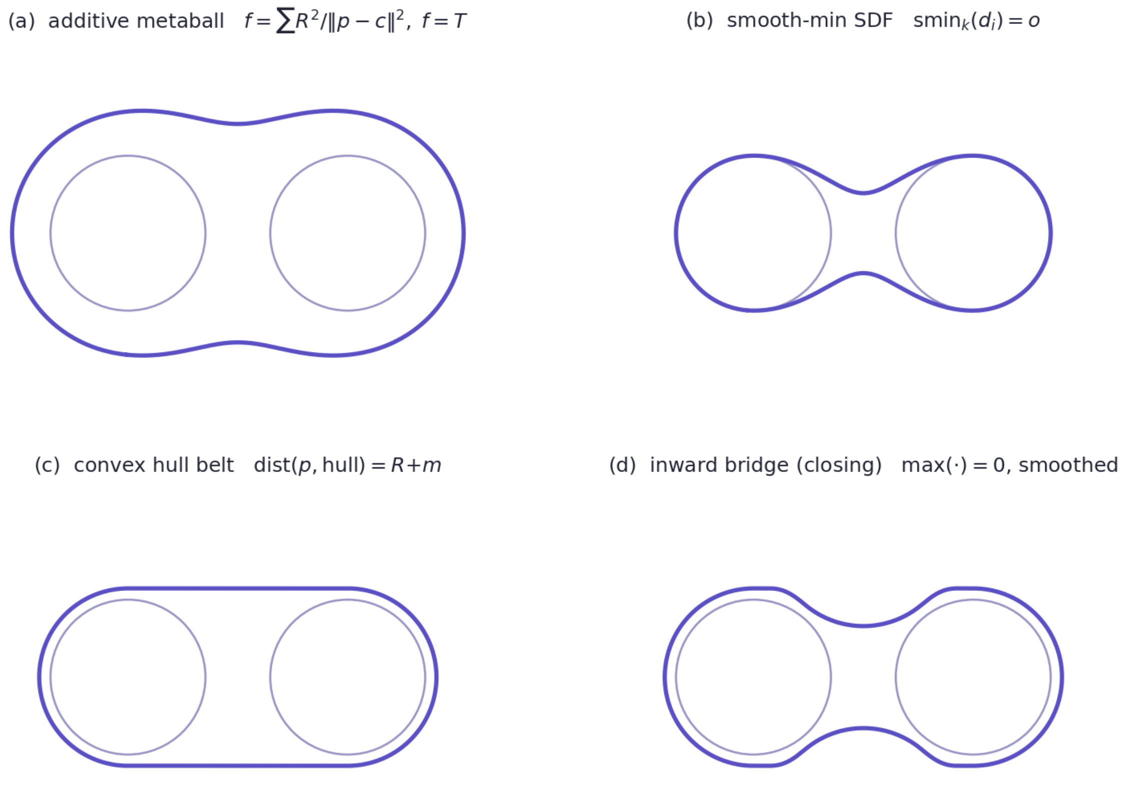


Figure 2: The four fields, all on the same two circles. (a) additive metaballs — a fat peanut with a soft neck; (b) smooth-min SDF — a tight, gently filleted neck; (c) convex-hull belt — semicircular caps joined by straight tangents, no inward pinch; (d) the morphological closing of §6 — the line wraps each circle, then the bridge curves inward.

6 Inward bridges via morphological closing

The belt's bridge is straight. To make it curve inward "as if it wrapped an invisible circle in the gap," we apply a **morphological closing** of the disks by a structuring disk of radius ρ,

$$S \bullet \mathcal{D}_\rho \;=\; \big(S \oplus \mathcal{D}_\rho\big) \ominus \mathcal{D}_\rho, \tag{8}$$

which fills any concavity narrower than $\sim 2\rho$ with a circular fillet of radius ρ. The rolling-ball picture is exact: a ball of radius ρ rolled around the *outside* cannot enter the narrow gap, so it bridges across and traces a concave arc of radius ρ — the invisible circle.

Placing the invisible circle. Let the cap radius be $R_c = R + m$. The phantom disk of radius ρ must be externally tangent to both caps,

$$\|P - c_A\| \;=\; \|P - c_B\| \;=\; R_c + \rho, \tag{9}$$

so P lies on the perpendicular bisector of $c_A c_B$, offset from the midpoint M by

$$\boxed{h \;=\; \sqrt{(R_c + \rho)^2 - \left(\tfrac{D}{2}\right)^2}}, \qquad D = \|c_B - c_A\|\,. \tag{10}$$

We then *carve* the phantom disks out of the belt. With $\Omega = \{p : f(p) \le 0\}$,

$$f(p) \;=\; \max\Big(\; \mathrm{dist}(p, \mathrm{hull}) - R_c,\;\; \max_g \big(\rho - \|p - P_g\|\,\big)\Big). \tag{11}$$

For a pair of circles there are two phantoms ($M \pm h\,\hat{n}$, top and bottom); for the triangle there is one per outer edge, with the sign of $\hat{n}$ chosen so that P lies away from the centroid. Figure 3 shows the construction.

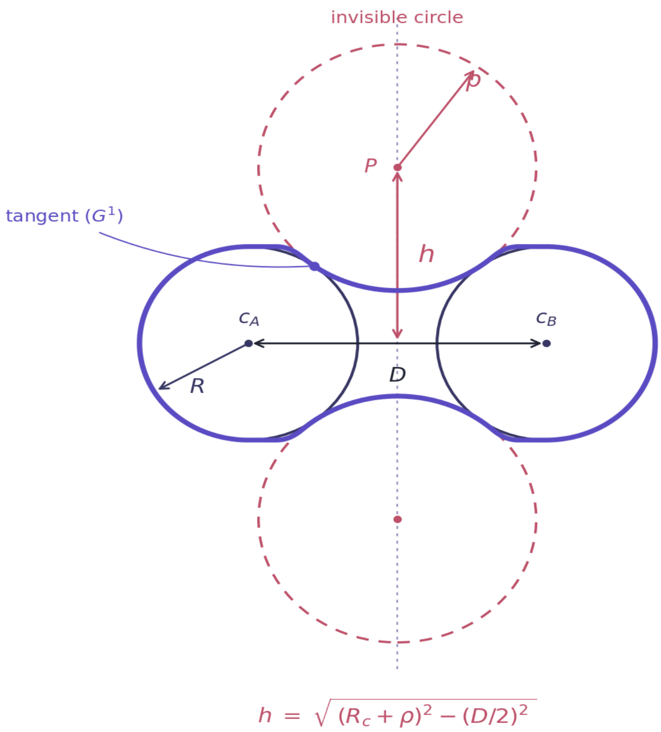


Figure 3: The carve, drawn with $m = 0$ for clarity. Each dashed circle is the rolling ball of radius ρ; its centre P sits at height h from (10) on the bisector, tangent to both circles. The bridge is the arc of the phantom facing the gap; with a coat m replace R by $R_c = R + m$.

Two limits. As $\rho \to \infty$, $h \to \infty$ and the fillet flattens back to the straight tangent — the taut belt is the $\rho = \infty$ case. As ρ shrinks, the bridge curls inward harder; the top and bottom phantoms (spaced $2h$ apart) overlap and *sever the neck* once $h \le \rho$, i.e.

$$\rho \;\le\; \frac{(D/2)^2 - R_c^2}{2R_c}. \tag{12}$$

7 Removing the cusps: a smooth Boolean

At a junction the wrap-arc (a cap, curvature $+1/R_c$) meets the bridge (the phantom, curvature $-1/\rho$). By construction the two circles are tangent there, so Γ is G^1; but the curvature flips sign instantaneously. That is a C^1 discontinuity in κ (cf. (2)), and on a sampling grid the hard max in (11) creates a crease that the discretised contour renders as a sharp point (Fig. 4, left).

The fix is to round the Boolean itself: replace max by the smooth maximum (6),

$$f(p) \;=\; \mathrm{smax}_s\Big(\,\mathrm{dist}(p,\mathrm{hull}) - R_c,\; \mathrm{smax}_{s\ ,g}\big(\rho - \|p - P_g\|\,\big)\Big). \tag{13}$$

Because smax_s differs from max only where the two fields lie within s of each other, it blends *only* in the narrow band at each junction, inserting a curvature ramp of width $\sim s$ and leaving the caps, the bridge, and the bridge depth untouched (Fig. 4, right). The trade-off: as s grows the junction tangency softens and the shape drifts back toward the smin blob, so one keeps $s \ll \rho, R$ (in practice $s \approx 10$–15 px against $R = 52$ killed the grid spike with no visible change to the geometry).

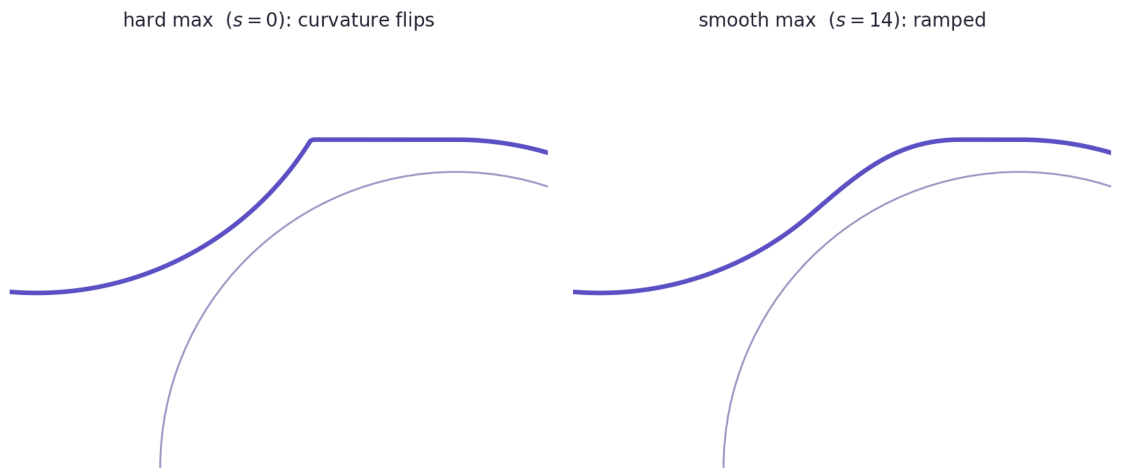


Figure 4: A junction, zoomed. Left: the hard intersection ($s = 0$) flips curvature in one step and shows a point. Right: the smooth-max ($s = 14$) ramps the curvature over a band of width $\sim s$, giving one continuous flowing line while the bridge still wraps the same circle.

8 From field to curve: the rendering pipeline

The field is sampled and contoured with **marching squares**. Each grid cell is classified by the signs of $f - T$ at its corners; on an edge between corner values v_a, v_b the crossing is the linear root

$$t \;=\; \frac{T - v_a}{v_b - v_a}, \tag{14}$$

and the two ambiguous saddle cases are resolved consistently. Because a crossing on a shared edge is computed from the *same* two corner values in both adjacent cells, the two cells emit identical endpoints, so the loose segments can be **chained** into ordered closed loops by hashing endpoints. Finally each loop is drawn as a quadratic-Bézier spline that passes through the edge *midpoints* $m_i = \frac{1}{2}(p_i + p_{i+1})$ using the sample points p_i as control points,

$$B_i(t) \;=\; (1-t)^2\, m_i \;+\; 2t(1-t)\, p_{i+1} \;+\; t^2\, m_{i+1}, \qquad t \in [0,1]. \tag{15}$$

Consecutive arcs share a tangent at every midpoint, so the rendered curve is C^1 and any residual faceting from the grid disappears.

9 Matching the Lightfall look

Two observations close the loop with the artwork.

- **Smoothness is a single dial.** Since $\kappa_{\max} \approx 1/(\text{blend size})$, a smoother line just means a larger blend. The screen's joins read as arcs several times the circle radius, so $k \approx 2R$ (smin) or a large ρ (closing) reproduces the gentle sweep.
- **The glow is the medium, not the geometry.** The luminous quality is thin, additive (screen-blended) strokes over a dark field — a rendering choice layered on top of the same curve, as in Fig. 5.

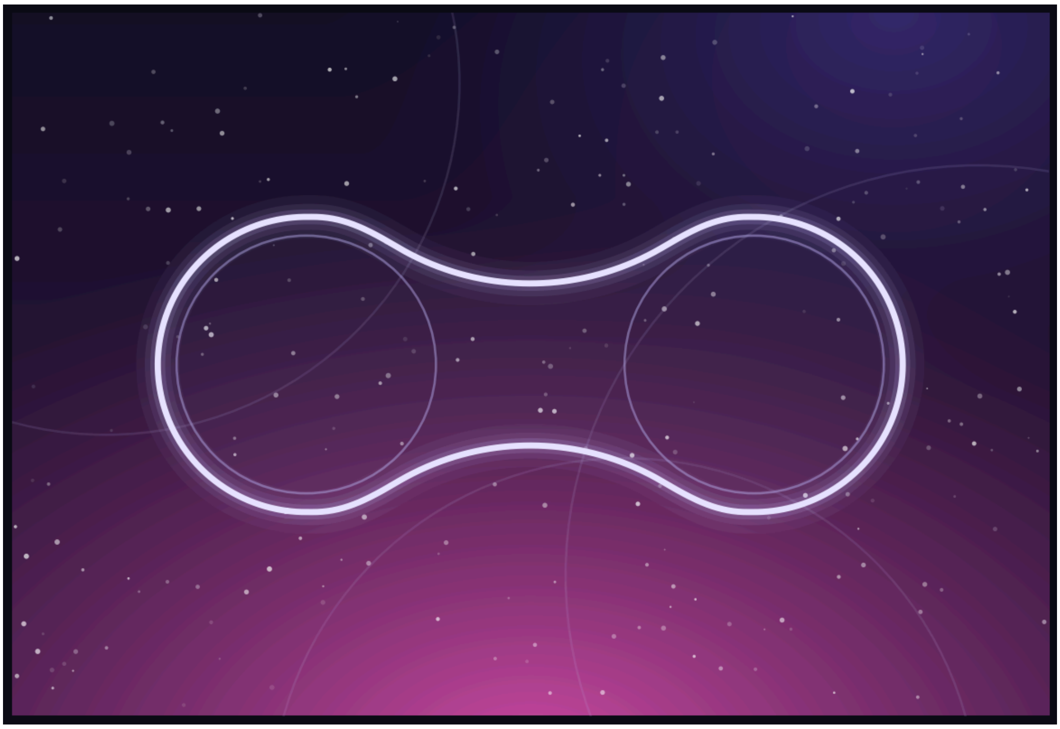


Figure 5: The two-circle closing (ρ finite, s small) rendered as a thin glowing stroke on a purple-magenta field with a starfield and faint background tracery — the same level set as Fig. 2d, in the title-screen medium.

10 Conclusion

The problem "smoothly enrobe these circles" is, underneath, a question about the smoothness of an implicit field. Four fields answer progressively sharper versions of it: additive metaballs give an organic but globally coupled blob; the smooth-min SDF makes the blend local and gently filleted; the convex hull gives a taut belt with semicircular caps; and the morphological closing bends each bridge inward around an exact rolling-ball arc. The recurring tool is the polynomial smooth min/max (5)–(6): hard min / max give exact constructive geometry but sharp curvature flips, and the $\pm\frac{1}{4}k\,h^2$ blend buys G^1 flow for negligible cost. Everything else — marching squares, loop chaining, the midpoint spline — is the bookkeeping that turns the field into a line.

A Reference kernels (pseudocode)

```
function smin(a, b, k): # quadratic smooth-min, C1
    if k <= 0: return min(a, b)
    h = max(1 - |a - b| / k, 0)
    return min(a, b) - 0.25 * k * h * h
```

```
function smax(a, b, k): # = -smin(-a, -b, k)
    if k <= 0: return max(a, b)
    h = max(1 - |a - b| / k, 0)
    return max(a, b) + 0.25 * k * h * h

function field(p): # inward-bridge closing, smoothed
    v = dist(p, hull_of_centers) - (R + m)
    for P in phantoms(centers, R, m, rho): # h = sqrt((R+m+rho)^2 - (D/2)^2)
        v = smax(v, rho - |p - P|, s)
    return v # outline is the level set v = 0
```