Construction A, Equal Radii

Build Procedures for $n = 2$ and $n = 3$ Circles

Overview

Construction A produces a single smooth outline enrobing a set of equal-radius circles by dilating the convex hull of their centres and carving concave bridges into the gaps. Both the two-circle and three-circle cases follow the same four-stage pipeline:

$$\textbf{hull} \rightarrow \textbf{phantom placement} \rightarrow \textbf{carve} \rightarrow \textbf{extract curve}.$$

What changes between $n = 2$ and $n = 3$ is the shape of the hull, how many phantom circles are placed, and one extra cleanup step needed only when three or more bridges are active simultaneously.

1 The $n = 2$ Case

1.1 Step 1 — Build the taut belt

The convex hull of two centres degenerates to the line segment $c_A c_B$. Dilating this hull by the coat radius $R_c = R + m$ gives a *stadium* shape:

$$\Gamma_{\text{belt}} = \{\, p : \operatorname{dist}(p, \text{hull}) = R_c \,\}.$$

This is two straight tangent segments, parallel to the centre line and offset by R_c, joined by two semicircular caps of radius R_c. It is already G^1: curvature is constant $(1/R_c)$ on the arcs and zero on the straight segments.

1.2 Step 2 — Place the phantom circle

A phantom circle of radius ρ is placed tangent to both coats:

$$\|P - c_A\| = \|P - c_B\| = R_c + \rho.$$

P lies on the perpendicular bisector of $c_A c_B$, at height

$$h = \sqrt{(R_c + \rho)^2 - \left(\tfrac{D}{2}\right)^2}, \qquad D = \|c_B - c_A\|,$$

from the midpoint M. There are two solutions, $P = M \pm h\hat{n}$. For $n = 2$ **both** are used, giving a bridge arc on each side of the line — the familiar peanut/dumbbell silhouette.

Pinch-off check. A valid bridge requires

$$\rho \;>\; \frac{(D/2)^2 - R_c^2}{2R_c}.$$

Below this threshold, the two phantoms (spaced $2h$ apart) overlap each other and the neck severs before the bridge can be drawn.

1.3 Step 3 — Carve the bridge

Each phantom disk subtracts from the belt via a hard maximum:

$$f(p) = \max\Big(\operatorname{dist}(p, \text{hull}) - R_c,\ \max_g \left(\rho - \|p - P_g\|\right)\Big).$$

Wherever a phantom disk overlaps the belt's interior, the field is pulled back toward zero, biting a concave arc out of each side of the gap.

1.4 Step 4 — Smooth the junction

The hard max in Step 3 creates a curvature discontinuity where the belt (curvature $+1/R_c$) meets the bridge (curvature $-1/\rho$) — visible as a crease under sampling. This is fixed by replacing the hard maximum with the quadratic smooth-maximum:

$$\operatorname{smax}(a, b, k) = \max(a, b) + \tfrac{1}{4}k\, h(a, b, k)^2, \qquad h(a, b, k) = \max\Big(1 - \tfrac{|a-b|}{k},\ 0\Big),$$

applied at blend width $s \ll \rho, R$:

$$f(p) = \operatorname{smax}_s\Big(\operatorname{dist}(p, \text{hull}) - R_c,\ \max_g \left(\rho - \|p - P_g\|\right)\Big).$$

This ramps the curvature transition over a narrow band of width $\sim s$ without altering the caps or bridge depth.

1.5 Step 5 — Extract the curve

The field is sampled on a grid and contoured with **marching squares**: each grid cell is classified by the sign of f at its corners, and zero-crossings on shared edges are computed once so adjacent cells agree, letting segments be **chained** into a single closed loop. The loop is then rendered as a **quadratic-Bézier spline** through the edge midpoints $m_i = \frac{1}{2}(p_i + p_{i+1})$, using the sample points as control points — giving a C^1 curve regardless of grid resolution. For $n = 2$ this reliably yields exactly one closed loop; no further filtering is needed.

2 The $n = 3$ Case

The pipeline is identical to $n = 2$ in every stage; two additions are required because three bridges are now active at once.

2.1 Step 1 — Build the taut belt

The convex hull of three centres is (generically) a **triangle**. Dilating it by R_c gives the Minkowski sum of the triangle with a disk of radius R_c: straight edges offset outward, joined at each vertex by a circular arc of radius R_c — a rounded triangle. The formula is unchanged from Step 1 of the $n = 2$ case; only the hull geometry differs.

2.2 Step 2 — Place phantoms, one per edge

For each of the three hull edges, the same tangency equations as before are solved. The key difference: for $n = 2$ both $\pm$ solutions are valid and both are used; for $n = 3$, using both is unsafe, since one candidate can land *inside* the triangle and carve into the cluster's interior rather than into an empty gap.

Side-selection rule. Only **one** phantom per edge is kept, chosen to lie away from the triangle's centroid. Formally, for edge (i, j), the signed perpendicular offset of every other circle's centre from the line $c_i c_j$ is averaged,

$$\bar{\sigma} = \frac{1}{n-2} \sum_{k \neq i,j} \sigma_k, \qquad \sigma_k = \left(c_k - \tfrac{c_i + c_j}{2}\right) \cdot \hat{n},$$

and the phantom candidate on the **opposite** side from $\bar{\sigma}$ is selected. For a triangle there is exactly one other circle per edge, so this reduces to simply placing P away from the third circle (equivalently, away from the centroid).

2.3 Step 3 — Carve all three bridges

The same carving formula as $n = 2$, extended to sum over all three active phantoms:

$$f(p) = \max\Big(\operatorname{dist}(p, \text{hull}) - R_c,\ \max_{g \in \{1,2,3\}} \big(\rho - \|p - P_g\|\big)\Big).$$

2.4 Step 4 — Smooth all junctions

The identical smax_s substitution from the $n = 2$ case is applied, now smoothing all six cap–bridge junctions (two per hull edge, three edges).

2.5 Step 5 — Extract the curve, with an artefact filter

Marching squares and the Bézier spline fit proceed exactly as in the $n = 2$ case. However, with three bridge carves active simultaneously, the fields can overlap near the triangle's **centroid** and produce a second, small, disconnected zero-crossing — a spurious loop distinct from the true outline.

Fix. Every closed loop returned by marching squares is measured by its number of sample points, and only the largest is kept:

$$\Gamma = \arg \max_{\gamma \in \{\text{contour loops}\}} |\gamma|.$$

This is a pragmatic filter, not a proven guarantee — correct in every configuration tested, with no formal argument that the artefact is always smaller than the true outline.

3 Summary

Step	$n = 2$	$n = 3$
Hull shape	segment → stadium	triangle → rounded triangle
Phantoms per gap	2 (both sides used)	1 per edge (outward side only)
Side selection	none needed (symmetric)	away-from-centroid rule
Carve & smooth	smax_s, one phantom term	smax_s, three phantom terms
Post-extraction	single loop, no filter needed	discard spurious loop, keep largest

The hull-dilation formula, the phantom tangency equations, the smooth-maximum kernel, and the marching-squares/Bézier extraction pipeline are shared, unmodified code in both cases. The only genuinely new logic introduced at $n = 3$ is the side-selection rule (before carving) and the largest-loop filter (after extraction).