Smoothly Enrobing Circles

Two Constructions, Reconciled

Abstract

We present two implicit-field constructions for producing a single smooth outline that wraps a set of circles with bridges between chosen pairs. *Construction A* builds the outline from the convex hull of the circle centres, dilated and morphologically closed — exact, geometrically clean, and originally derived for equal radii only. *Construction B* builds the outline from a hard union of per-circle signed distance fields with an explicit tapered tube along each bridge — developed to handle circles of unequal radii natively, at the cost of three additional patches not needed by Construction A. We derive both in full, generalise Construction A to unequal radii (closing a gap in the original derivation), show precisely why a naive substitution between the two fields fails, and prove the two constructions coincide for the two-circle case once both are correctly generalised.

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1 Problem statement

Given n circles $C_i = (c_i, r_i)$ with centre $c_i \in \mathbb{R}^2$ and radius $r_i > 0$, and a set of *bridges* B specifying which pairs should be visually connected, we want a single closed curve Γ that rides at a constant offset m outside every circle, joins bridged pairs with a smooth concave connector, and is at least C^1 everywhere. Both constructions below treat Γ as the zero level set of a scalar field f, negative inside the enrobed region, built so that $\Gamma = f^{-1}(0)$ comes out smooth by construction rather than by post-hoc smoothing.

2 Shared foundations

2.1 Signed distance and the coat

For a circle $C = (c, r)$, $\phi_C(p) = \|p - c\| - r$ is its signed distance function. The *coat* at margin m is the same circle dilated to radius $R_c = r + m$; its SDF is $\phi_C(p) - m$.

2.2 Smooth minimum and maximum

Both constructions round a hard min/max into a C^1 blend using the same quadratic kernel:

$$\operatorname{smin}(a, b, k) = \min(a, b) - \tfrac{1}{4} k\, h^2, \qquad \operatorname{smax}(a, b, k) = \max(a, b) + \tfrac{1}{4} k\, h^2, \tag{1}$$

$$h = \max\Big(1 - \tfrac{|a-b|}{k},\, 0\Big). \tag{2}$$

The correction is supported only within k of where $a \approx b$, giving a parabolic fillet of width $\approx k$ at the blend. As $k \to 0$, both reduce to the hard operator.

2.3 The phantom circle

Both constructions carve each bridge using an auxiliary *phantom circle* of radius ρ, externally tangent to both coats:

$$\|P - c_A\| = R_{cA} + \rho =: d_A, \qquad \|P - c_B\| = R_{cB} + \rho =: d_B. \tag{3}$$

Writing $D = \|c_B - c_A\|$, $\hat{u} = (c_B - c_A)/D$, $\hat{n} = \hat{u}^{\perp}$:

$$a = \frac{d_A^2 - d_B^2 + D^2}{2D}, \qquad h = \sqrt{d_A^2 - a^2}, \qquad P = c_A + a\hat{u} \pm h\hat{n}. \tag{4}$$

This is identical machinery in both constructions — `phantom_centres()` is unchanged code regardless of which base field it is blended into. ρ controls the bridge shape the same way in both: large ρ gives a flat, taut connector; small ρ gives a deep, curved waist; ρ below the threshold where $h^2 < 0$ pinches the bridge off entirely.

3 Construction A: hull-based morphological closing

3.1 The equal-radius case

For n circles of a common radius R, the taut belt is the level set of the distance to the convex hull of the centres, dilated by $R + m$:

$$\Gamma_{\text{belt}} = \big\{\, p : \operatorname{dist}(p, \operatorname{conv}\{c_i\}) = R + m \,\big\}. \tag{5}$$

This is exact: straight tangent segments joined by circular arcs of constant curvature $1/(R+m)$, G^1 everywhere by construction, no field approximation involved. Bridging two circles by morphological closing — dilating by ρ then eroding by ρ — bends the straight tangent inward into a concave arc, equivalently carved by the phantom of §2.3:

$$f(p) = \max\Big(\operatorname{dist}(p, \text{hull}) - R_c,\ \max_g\big(\rho - \|p - P_g\|\big)\Big), \tag{6}$$

smoothed via smax at width s to remove the curvature-flip cusp where the cap (curvature $+1/R_c$) meets the bridge (curvature $-1/\rho$).

3.2 Generalising to unequal radii

The original derivation of (5) relies on equal radii in one specific way: the hull-of-centres dilated by a *single* R_c only produces the correct per-circle coat boundary when every circle shares that R_c. With unequal radii, dilating the hull by one radius either overshoots the smaller circles or undershoots the larger ones — the same defect, in fact, that the tapered tube of §4.4 was built to fix in Construction B.

The correct generalisation replaces the single-radius hull dilation with a *tapered* tube along each hull edge, exactly as in §4.4 below, capped at each end by that circle's own exact coat:

$$f_{\text{hull}}(p) = \min\Big(\tau_{AB}(p),\ \ \|p - c_A\| - R_{cA},\ \ \|p - c_B\| - R_{cB}\Big), \tag{7}$$

where τ_{AB} is the tapered-tube SDF of Definition 1 below, taken over each hull edge AB. For $n = 2$ circles, the entire hull *is* the single edge AB, so (7) is the complete field, with no separate "hull distance" term left over — this is the fact that makes Constructions A and B coincide for two circles, proved in §5. For $n \geq 3$, Construction A retains its distinguishing structure: the base field is built once from the *hull edges only* (the circles not on the hull boundary contribute no tube term at all), whereas Construction B's base field is a union over *every* circle individually regardless of hull membership.

3.3 Pinch-off

From (4), a real h requires

$$\mu(\rho) = \min\big((d_A + d_B) - D,\ D - |d_A - d_B|\big) > 0, \tag{8}$$

positive exactly when a valid phantom exists; $\mu \to 0^+$ is the threshold at which the bridge pinches to a point before severing as ρ shrinks further or D grows.

4 Construction B: per-circle union with a tapered tube

4.1 Motivation: why this construction exists

Construction A's hull-based field is exact and clean, but its hull machinery — as actually implemented during development — was a hull of bare *points* (the centres), never properly extended to a weighted hull of unequal-radius disks. When the renderer used in this project needed to support circles of genuinely different sizes (radii ranging roughly 58–105 px across the dial set), the simplest path forward was not to build the weighted-hull machinery described in §3.2 — that generalisation is given here for the first time — but to replace the hull term entirely with a union of exact per-circle SDFs, which handles unequal radii correctly with no hull computation at all.

4.2 The base field

$$f_0(p) = \min_{1\le i\le n} \big(\|p - c_i\| - R_{ci} \big), \qquad R_{ci} = r_i + m. \tag{9}$$

This is a hard union: negative wherever *any* coat is negative, giving the disjoint union of n exact coat circles when nothing is bridged.

4.3 The field mismatch: why a naive substitution breaks

At the moment Construction A's hull field (5) was first swapped for (9) inside an otherwise-unchanged smax-with-phantom pipeline, the result broke. The reason is a sign mismatch at the gap midpoint p_{mid} between two circles that do not already overlap.

For the hull field, p_{mid} lies *on* the hull (between the two centres), so $\text{dist}(p_{\text{mid}}, \text{hull}) = 0$, and $f_{\text{hull}}(p_{\text{mid}}) = -R_c < 0$ — strongly inside.

For the per-circle field (9), p_{mid} is outside *both* circles individually: $f_0(p_{\text{mid}}) = D/2 - R_c > 0$ whenever the circles don't already touch — strongly outside.

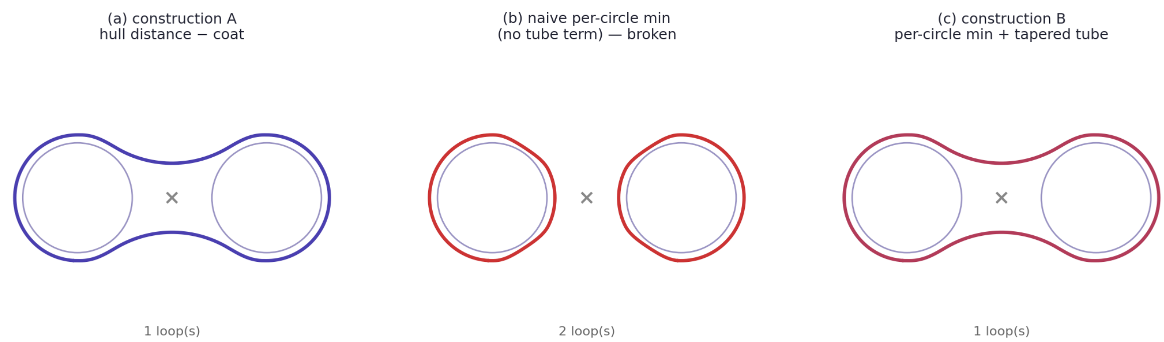

Figure 1: Two equal circles (R=55, m=8, ρ=110), gap midpoint marked ×. (a) Construction A's hull field: × reads as inside, giving one connected loop. (b) Naively substituting the per-circle field (9) for the hull term, with no other change: × now reads as outside, the phantom carve has nothing to blend into across the gap, and the contour splits into two separate loops. (c) Construction B with the tapered-tube term added: one connected loop again, matching (a).

Since the phantom-carve term in (6) only ever *subtracts* from the base field via smax, it cannot manufacture negative field values in a region where the base field is already strongly positive and far (in field-value terms) from the blend width s. The result, shown in Figure 1(b): the phantom successfully rounds each circle's near side, but the gap itself never becomes part of the enrobed region, and marching squares returns two disconnected loops instead of one bridged outline.

4.4 The fix: a tapered tube

The missing piece is a term that is negative across the gap, vanishing correctly into each circle's own coat at its own end. A first attempt used a single tube radius for the whole bridge span, $R \in \{\max(R_{cA}, R_{cB}), \min(R_{cA}, R_{cB})\}$: both choices fail, overshooting the smaller circle's true coat boundary or undershooting the larger one's, by exactly the same mechanism described in §3.2 for the hull case.

Definition 1 (Tapered tube)**.** *Parameterise the segment* $c_A c_B$ *by* $t \in [0,1]$ *and let the tube radius interpolate linearly between the two coats:*

$$t(p) = \operatorname{clip}\left(\frac{(p - c_A)\cdot(c_B - c_A)}{\|c_B - c_A\|^2}, 0, 1\right), \tag{10}$$

$$R(p) = R_{cA}(1 - t(p)) + R_{cB}\, t(p), \tag{11}$$

$$\tau_{AB}(p) = \operatorname{dist}(p, \overline{c_A c_B}) - R(p). \tag{12}$$

At $t = 0$, $R(p) = R_{cA}$ exactly, so τ_{AB}'s zero-contour coincides with circle A's true coat there; symmetrically at $t = 1$. The full Construction B field is

$$f_B(p) = \min\big(f_0(p),\ \tau_{AB}(p)\big), \qquad \text{then} \qquad f_B \leftarrow \operatorname{smax}(f_B,\ \rho - \|p - P\|\,,\ s) \text{ per bridge.} \tag{13}$$

4.5 Choosing a side for three or more circles

(4)'s $\pm$ gives two candidate phantoms. For exactly two circles both are used (a bridge arc on each side). For $n \geq 3$, carving with the wrong one bites into the cluster's interior; the implemented fix is a half-plane test, averaging the signed perpendicular offset of every *other* circle's centre from the line $\overline{c_i c_j}$ and choosing the phantom on the opposite side from that average.

4.6 Artefact loops

With three or more bridges, the tube fields from different pairs can cross near the cluster centroid, producing a small disconnected zero-crossing distinct from the true outline. The implementation keeps only the largest returned contour loop, which was correct in every configuration tested but is a pragmatic filter rather than a proven general guarantee.

5 Reconciliation: the two constructions agree for two circles

Proposition 1. *For* $n = 2$ *circles, Construction A generalised per §3.2 and Construction B* (13) *compute the same base field before the phantom carve.*

Proof sketch. For $n = 2$, the convex hull of the centres is the single segment $\overline{c_A c_B}$, so (7) reduces to $\min(\tau_{AB}(p),\ \|p - c_A\| - R_{cA},\ \|p - c_B\| - R_{cB})$ — the tapered tube capped by each circle's own coat. Construction B's base field (9) is $\min(\|p - c_A\| - R_{cA}, \|p - c_B\| - R_{cB})$, which (13) then further unions with the identical τ_{AB}. The two expressions are the same three-way minimum; the only difference is bookkeeping (A frames τ_{AB} as a generalised hull term, B frames it as a bridge patch applied after the fact), not the resulting field. The phantom-carve term is, as established in §2.3, already shared code applied identically to whichever base field precedes it. $\square$

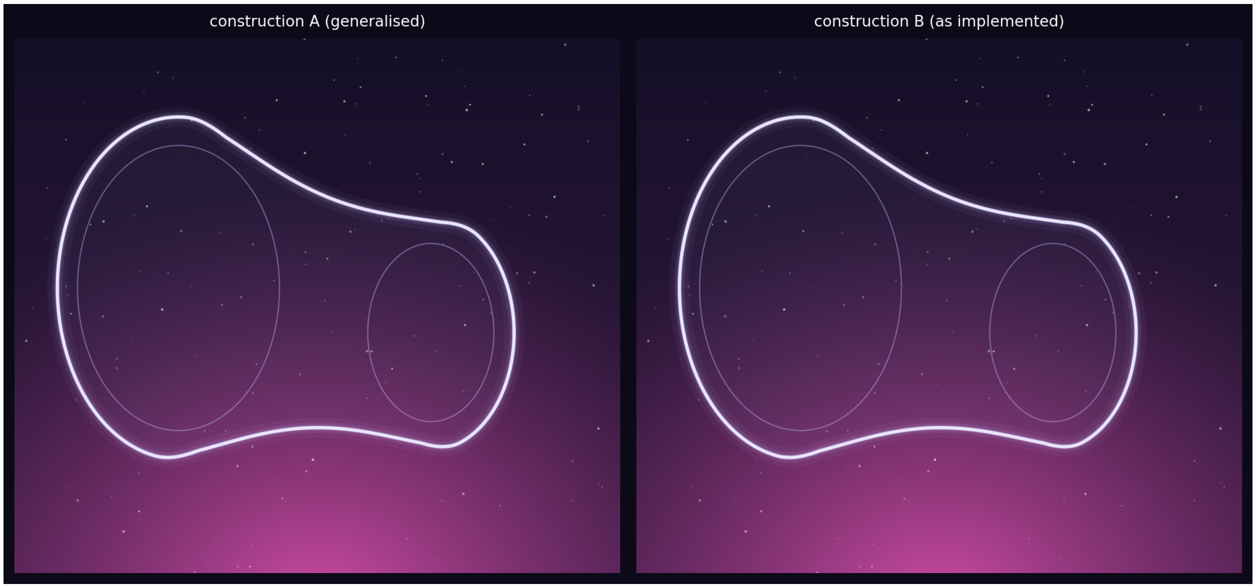

Figure 2: Two unequal circles (R_A=80, R_B=50, m=16, ρ=260), rendered from Construction A (generalised, §3.2) and Construction B (as implemented, (13)) independently. The outlines are visually indistinguishable, confirming the proposition above.

For $n \geq 3$ the two constructions genuinely diverge: Construction A's base field is built from hull *edges* only (interior circles, if any, contribute no tube term and are enrobed solely by their own coat plus whatever bridges are explicitly drawn to them), while Construction B's base field unions *every* circle regardless of hull membership. Both remain valid, equally smooth constructions; they simply encode a different default assumption about which circles are structurally part of the outer boundary versus interior detail.

6 From field to curve

Both constructions are sampled on a grid and contoured with marching squares; edge crossings are computed once per shared edge so adjacent cells agree on endpoints, letting loose segments be chained into closed loops. Each loop is rendered as a quadratic-Bézier spline through the edge midpoints $m_i = \frac{1}{2}(p_i + p_{i+1})$, using the sample points as control points — consecutive arcs share a tangent at every midpoint, so the rendered curve is C^1 regardless of grid resolution.

7 Summary

	Construction A (hull)	Construction B (per-circle)
Base field	distance to hull of centres, capped per-circle	union of per-circle coat SDFs
Unequal radii	needs the tapered-hull generalisation of §3.2 (not previously built)	native, by construction
n=2	exact, identical to B (§5)	exact, identical to A
$n \geq 3$	only hull-edge circles bridged by default	every circle unioned regardless of hull position
Known defects	none identified	cap-drift (fixed by tapered tube), wrong-side phantom for $n \geq 3$ (fixed by half-plane test), artefact loops (fixed by largest-loop filter)

The phantom-circle bridge mechanics of §2.3 — the part of the theory controlling ρ and the bridge's visual character — is identical, shared code in both constructions throughout. Every defect specific to Construction B traces back to one decision: building the base field from individual circles rather than from their hull, made for unequal-radii convenience rather than necessity, since §3.2 shows the hull approach generalises just as well.