Analytic Continuation

Differentiable many times at all points (for power series to exist)

Note : Analytic function : Function that is locally given by a convergent power series.

$\mathbb{C}$ (complex analytic)

Holomorphic function : It is a complex differentiable function.

Cauchy-Riemann Equations.

A function ~~f:ℝ~~ $f : \mathbb{R} \to \mathbb{R}$ is said to be differentiable at $x \in \mathbb{R}$ if

$$\lim_{\substack{h \in \mathbb{R} \\ h \to 0}} \frac{f(x+h) - f(x)}{h} \quad \text{exists.}$$

If the limit exists, its value is called $f'(x)$.

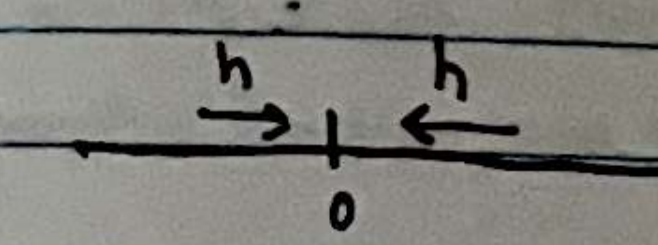
Analogously, a function $f : \mathbb{C} \to \mathbb{C}$ (ie. f takes a complex number and returns a complex number) is said to be complex differentiable at $z \in \mathbb{C}$ if

$$\lim_{\substack{h \in \mathbb{C} \\ h \to 0}} \frac{f(z+h) - f(z)}{h} \quad \text{exists.}$$

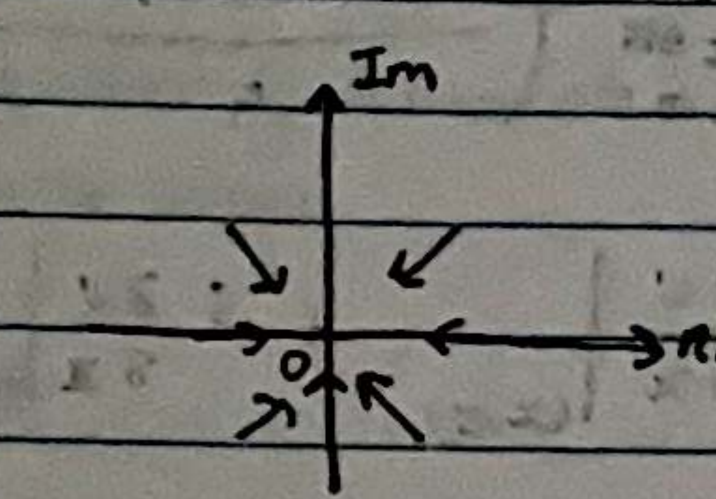
Again if limits exists, its value is called $f'(z)$. If f is complex differentiable at every $z \in U \subset \mathbb{C}$, then f is said to be holomorphic on U.

The complex differentiability condition is much stronger than $\mathbb{R} \to \mathbb{R}$ as when h approaches zero in the complex plane it must do so in all directions (there are infinitely many.) In real limit, there are only 2 directions from which h may approach zero (+ve & -ve directions)

ie. $\mathbb{R}$



$\mathbb{C}$



$\mathbb{C}$

Thus, if $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable at $z \in \mathbb{C}$ then

$$\lim_{\substack{t \in \mathbb{R} \\ t \to 0}} \frac{f(z+th) - f(z)}{th} \quad \text{exists for all } h \in \mathbb{C} \text{ and equals to } f'(z)$$

One can view $\mathbb{C}$ as $\mathbb{R}^2$ by identifying the complex number $x+yi$ with the pair (x,y)

we may thus write :

$f(z) = f(x,y) = u(x,y) + i.v(x,y)$, where $u(x,y) = Re(f(z))$ and $v(x,y) = Im(f(z))$

· But the complex differentiability condition requires

$$f'(z) = \lim_{\substack{t \in \mathbb{R} \\ t \to 0}} \frac{f(z+t\cdot 1) - f(z)}{t \cdot 1}$$

$$= \lim_{\substack{t \in \mathbb{R} \\ t \to 0}} \left(\frac{u(x+t,y) - u(x,y)}{t} + i \cdot \frac{v(x+t,y) - v(x,y)}{t} \right)$$

$$= \left.\frac{\partial u}{\partial x}\right|_{(x,y)} + i \left.\frac{\partial v}{\partial x}\right|_{(x,y)}$$

and

$$f'(z) = \lim_{\substack{t \in \mathbb{R} \\ t \to 0}} \frac{f(z+t\cdot i) - f(z)}{t \cdot i}$$

$$= \cancel{-i \lim_{t \in \mathbb{R}}}$$

$$-i \cdot \lim_{\substack{t \in \mathbb{R} \\ t \to 0}} \left(\frac{u(x,y+t) - u(x,y)}{t} + i \, \frac{v(x,y+t) - v(x,y)}{t} \right)$$

$$= -i \cdot \left(\frac{\partial u}{\partial y} + i \cdot \frac{\partial v}{\partial y} \right)$$

$$= \left.\frac{\partial v}{\partial y}\right|_{(x,y)} - i \cdot \left.\frac{\partial u}{\partial y}\right|_{(x,y)}$$

Setting these equal ~~give~~ prove that whenever f is complex-differentiable, it satisfies

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

These are the Cauchy-Riemann equations, a system of D.E that every holomorphic function must satisfy on its domain.

Analiticity of holomorphic functions

Any holomorphic function has an expansion as a power series.

Suppose $f : U \to \mathbb{C}$ is holomorphic with U an open set. Then there exists a sequence of complex numbers $\{a_n\}_{n \geq 0}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Within the radius of convergence about z_0 (i.e. it converges for all z in some disk centered at z_0 and equals $f(z)$ when it converges. Specifically,

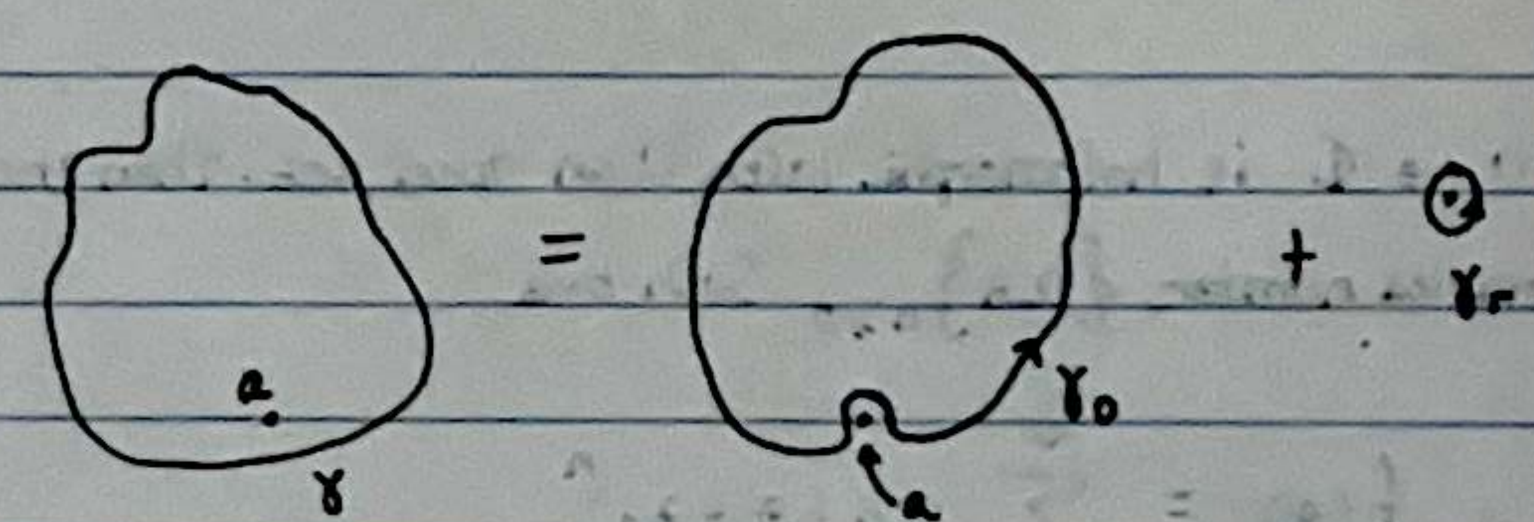
$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

Cauchy Integral Formula

It states that the values of a holomorphic function inside a disk are determined by the values of that function on the boundary of the disk. More precisely, Suppose $f: U \to \mathbb{C}$ is holomorphic and γ is a circle contained in U, Then for any a in the disk bounded by γ,

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz$$

More generally, γ is the boundary of any region whose interior contains a.



Cauchy differentiation formula

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz$$

If one knows the value of $f(z)$ on some closed curve γ, then one can compute the derivatives of f inside the region bounded by γ via an integral.

The formula can be proven by induction on n.

The case $n=0$ is simply the Cauchy integral formula.

Suppose the differential formula holds for $n=k$, using differentiation under the integral, we have

$$f^{(k+1)}(a) = \frac{d}{da} f^{(k)}(a)$$

$$= \frac{k!}{2\pi i}\int_\gamma \frac{d}{da}\frac{f(z)}{(z-a)^{k+1}}\,dz$$

$$= \frac{k!}{2\pi i}\int_\gamma \frac{(k+1)\,f(z)}{(z-a)^{k+2}}\,dz$$

$$= \frac{(k+1)!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{k+2}}\,dz$$

$\blacksquare$

One can use the cauchy integral formula to compute the contour integrals which take the form given in the integrand of the formula.

Ex.1 Compute $\int_C \frac{(z-2)^2}{z+i}\,dz$ where C is the circle of radius 2 centred at the origin.

Let $f(z) = (z-2)^2$; f is holomorphic everywhere in the interior of C. Hence, by the cauchy integral formula

$$\int_C \frac{(z-2)^2}{z+i}\,dz = 2\pi i\, f(-i) = -8\pi + 6\pi i$$

The Cauchy differentiation formula can also be used to evaluate integrals.

Compute: $\int_C \frac{z+1}{z^4+2z^3}\,dz$ where C is a circle of radius 1 centred at the origin.

Let $g(z) = \frac{z+1}{z+2}$; g is holomorphic everywhere inside C. Hence,

$$\int_C \frac{z+1}{z^4+2z^3}\,dz = \frac{2\pi i}{2!} g''(0) = -\frac{\pi i}{4}$$

Liouville's Theorem

If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$, then f is constant.

Statement: If a function is holomorphic everywhere in $\mathbb{C}$ and is bounded, then the function must be constant.

Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic. Futhermore, assume that

$$|f(z)| \leq 5|z| \quad \forall z \in \mathbb{C}. \text{ If } f(1) = 3+4i, \text{ what is } f(1+i)?$$

We consee: $\left|\frac{f(z)}{z}\right| \leq 5$, By Liouville's theorem, we can see that $\frac{f(z)}{z}$ is constant.

$\therefore$ Substituting

$$\frac{3+4i}{1} = \frac{f(1+i)}{1+i}$$

$$\therefore f(1+i) = 7i - 1 \leftarrow$$

or Since $f: \mathbb{C} \to \mathbb{C}$ is a holomorphic function and $|f(z)| \leq 5|z|$.

$\Rightarrow f(0) = 0$, the power series of f around $z = 0$ is :

$$f(z) = \sum_{k=1}^{\infty} \frac{f^{(k)}(z)}{k!} z^k$$

$\Rightarrow$ the function $g: \mathbb{C} \to \mathbb{C}$ such that $g(z) = \frac{f(z)}{z}$ $\forall z \in \mathbb{C}$ is a

bounded holomorphic function due to Liouville's Theorem $g(z) = C$ within C a constant. This implies that,

$$g(z) = \frac{f(1)}{1} = 3+4i, \ \forall z \in \mathbb{C} \Rightarrow f(1+i) = (3+4i)(1+i)$$

$$= -1 + 7i \leftarrow$$

Cauchy - Hadamard Theorem

Statement : Consider the formal power series in one complex variable z of the form

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad \text{where } a, c_n \in \mathbb{C}$$

The radius of convergence of f at the point a is given by

$$\frac{1}{R} = \limsup_{n \to \infty} \left(|c_n|^{1/n} \right)$$

Limit superior (limit as n approaches ∞ of the supremum) of the sequence values after the n^{th} position.

If the sequence values are unbounded so that lim sup is ∞, the power series does not converge near a, while if lim sup is 0 then the radius of convergence is ∞ meaning the series converges on the entire plane.

Without loss of generality, assume $a = 0$. We will show first that the power series $\sum c_n z^n$ converges for $|z| < R$, and then that it diverges for $|z| > R$.

First suppose $|z| < R$. Let $t = 1/R$ not be zero or $\pm$ infinity.

For any $\varepsilon > 0$, there exists only a finite number of n such that

$\sqrt[n]{|c_n|} \geq t + \varepsilon$. Now $|c_n| \leq (t+\varepsilon)^n$ for all but a finite number of n,

So the series $\sum c_n z^n$ converges if $|z| < 1/(t+\varepsilon)$. This proves the 1st part.

Conversely, for $\varepsilon > 0$, $|c_n| \geq (t-\varepsilon)^n$ for infinitely many c_n, so if

$|z| = \frac{1}{(t-\varepsilon)} > R$ we can see that the series cannot converge because its

n^{th} term does not tend to 0.

Analytic Continuation (Extension of the domain of an analytic function)

Consider a particular analytic function f. It is given by ~~a~~ a power series centered at $z=1$.

$$f(z) = \sum_{k=0}^{\infty} (-1)^k (z-1)^k$$

By the Cauchy-Hadamard Theorem, its radius of convergence is 1. That is f is defined and analytic on the open set $U = \{|z-1|<1\}$ which has boundary $\partial U = \{|z-1| = 1\}$. Indeed, the series diverges at $z=0 \in \partial U$.

Let us re centre the power series at a different point $a \in U$

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k.$$

We will calculate the a_ks and determine whether this new power series converges in an open set V which is not contained in U. ~~So~~ If so, we will have analytically continued f to the region $U \cup V$ which is strictly larger than U.

The distance a to ∂U is $\rho = 1 - |a-1| > 0$. Take $0<r<\rho$; let D be disk of radius r around a, and let ∂D be its boundary. Then $D \cup \partial D \subset U$

(But what to do when at singularities eg 0 for $\frac{1}{x}$?)

(Analytic continuation for real valued functions.)

Using Cauchy's differentiation formula to calculate the new coefficients:

$$a_k = \frac{f^{(k)}(a)}{k!} = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{k+1}} = \frac{1}{2\pi i}\int_{\partial D} \frac{\sum_{n=0}^{\infty}(-1)^n(\zeta-1)^n\,d\zeta}{(\zeta-a)^{k+1}}$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(-1)^n \int_{\partial D} \frac{(\zeta-1)^n\,d\zeta}{(\zeta-a)^{k+1}} = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(-1)^n \int_0^{2\pi} \frac{(a+re^{i\theta}-1)^n rie^{i\theta}\,d\theta}{(re^{i\theta})^{k+1}}$$

$$= \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n \int_0^{2\pi} \frac{(a-1+re^{i\theta})\,d\theta}{(re^{i\theta})^k}$$

$$= \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n \int_0^{2\pi} \frac{\sum_{m=0}^{n}\binom{n}{m}(a-1)^{n-m}(re^{i\theta})^m}{(re^{i\theta})^k}$$

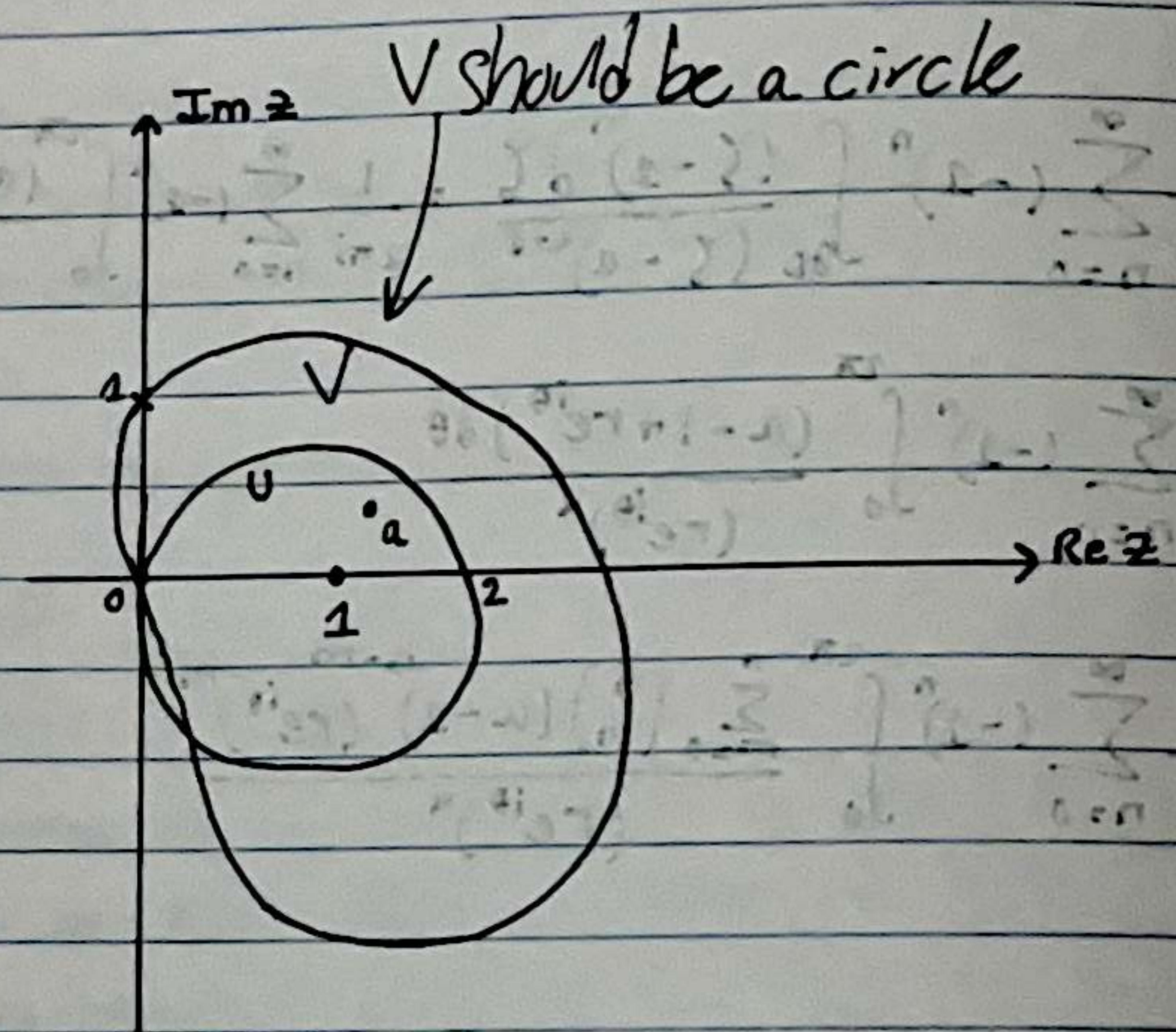
$\vdots$

$$= (-1)^k a^{-k-1}$$

That is: $f(z) = \sum_{k=0}^{\infty} a_k(z-a)^k = \sum_{k=0}^{\infty}(-1)^k a^{-k-1}(z-a)^k = \frac{1}{a}\sum_{k=0}^{\infty}\left(1-\frac{z}{a}\right)^k$

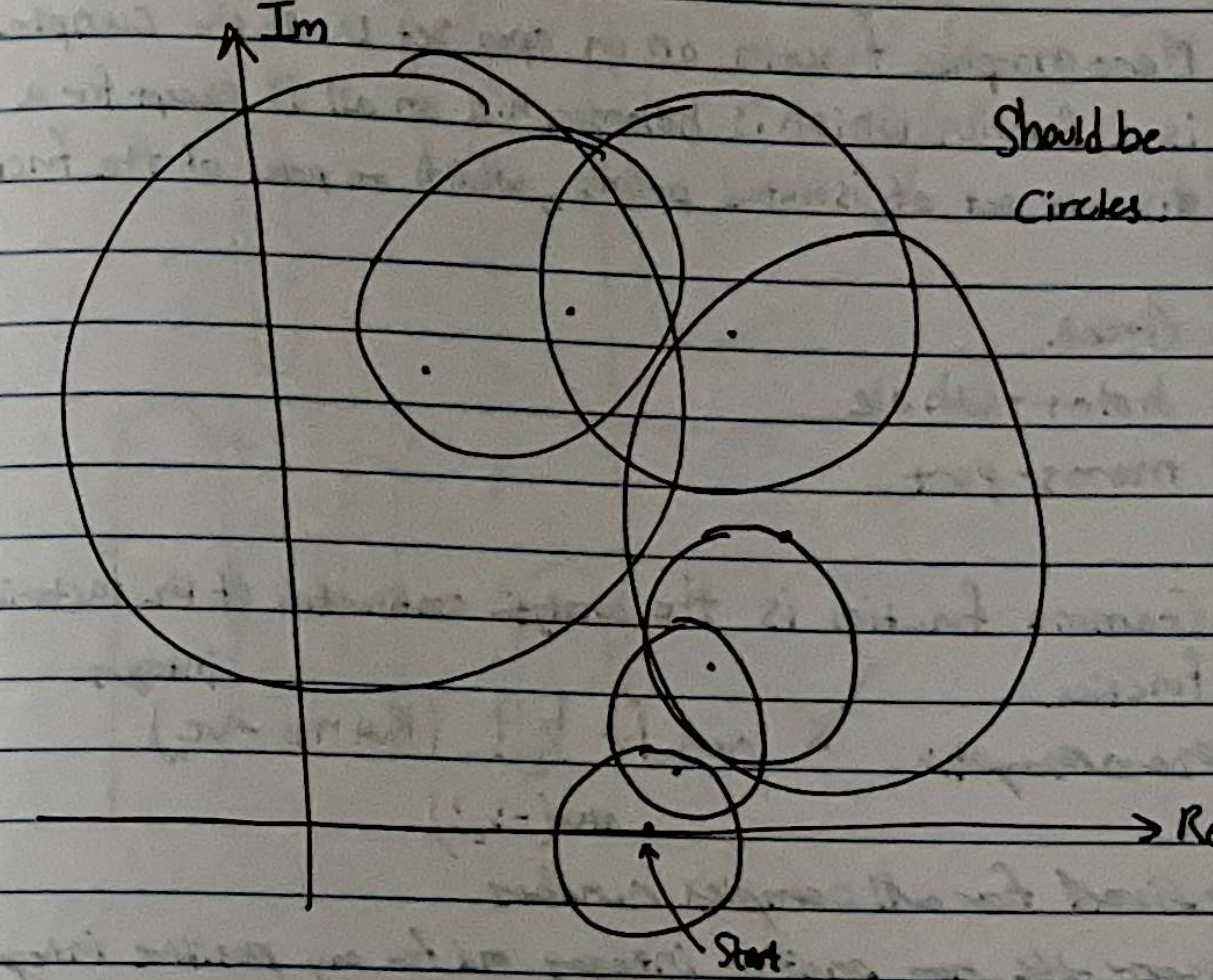
which has radius of convergence $|a|$, and $V = \{|z-a|<|a|\}$.

If we choose $a \in U$ with $|a| > 1$, then V is not a subset of U and is actually larger in area than U. The plot shows the result for $a = \frac{(3+i)}{2}$



We can continue this process: Select $b \in U \cup V$, recentre the power series at b and determine whether the new power series converges. If the region contains points not in $U \cup V$, then we will have analytically continued f even farther. This particular f can be analytically continued to the punctured complex plane $\mathbb{C} \setminus \{0\}$

Visualisation of analytic Continuation



Identity Theorem

~~It states~~: given

The identity theorem for holomorphic functions states: given functions f and g holomorphic over a domain D (open and connected subset), if $f = g$ on some $S \subseteq D$, S having $\underbrace{\text{a limit}}_{\text{an accumulation point}}$, then $f = g$ on D.

Meromorphic function on an open set D of the complex plane is a function which is holomorphic on all D except for a discrete set of isolated points, which are poles of the function.

Greek
holos - whole
meros - part

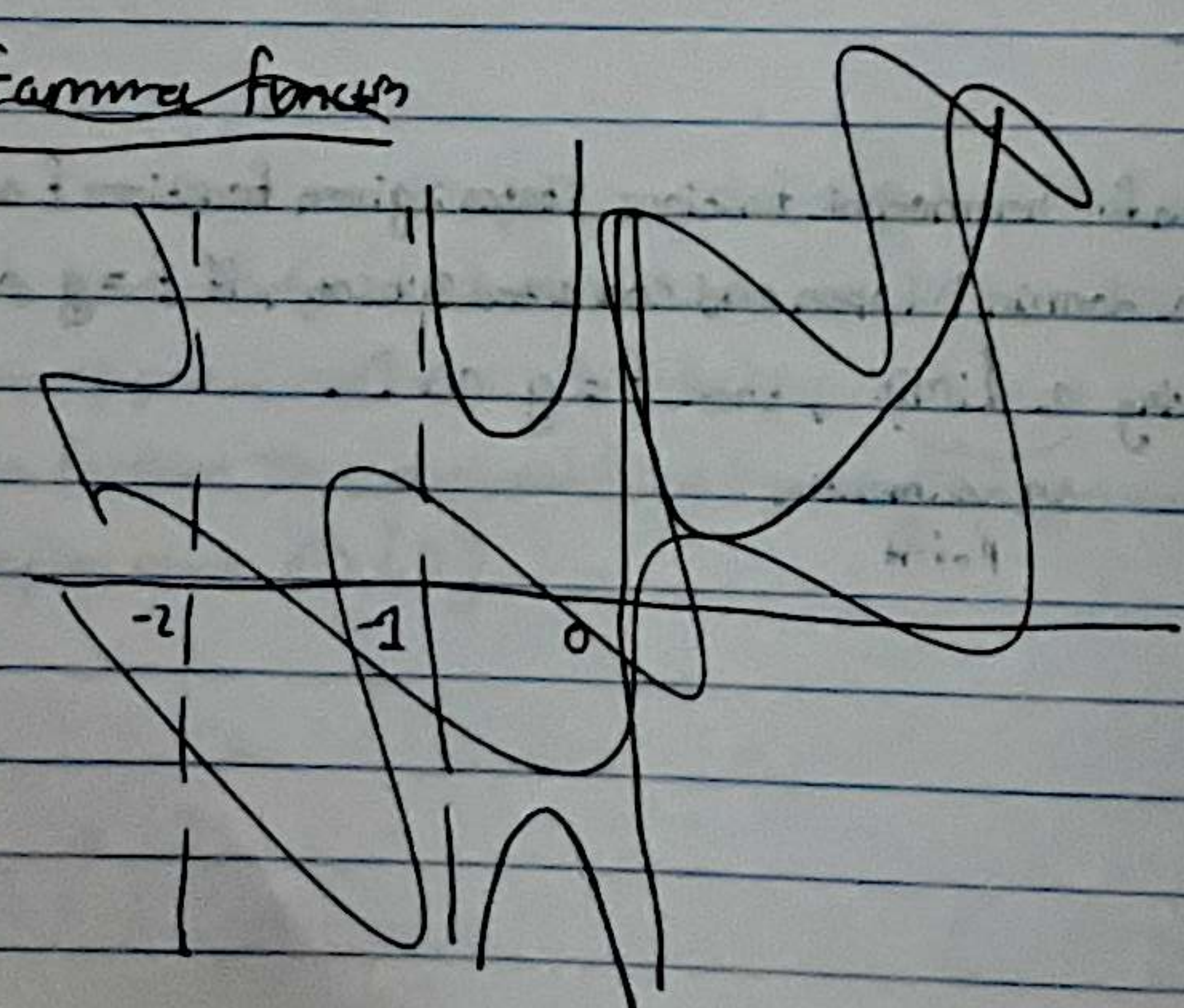
Gamma function is the analytic continuation of the factorial function.

meromorphic

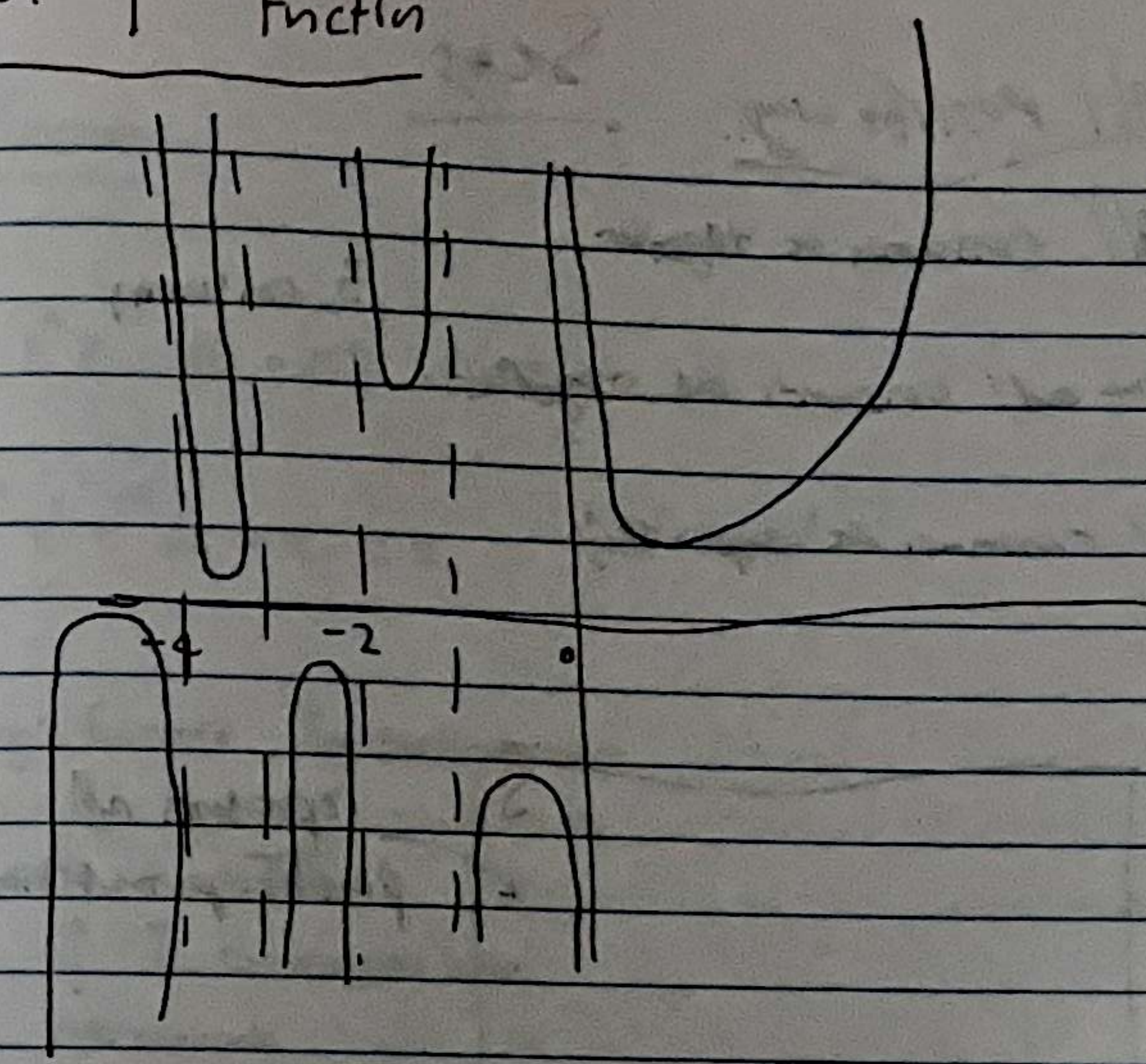
used for $\frac{1}{2}!$ (But no -ve integers) also $(-\frac{1}{2}!)$

defined for all complex numbers except the non positive integers and for any positive integer n.

Gamma function



Plot of Γ - function



Γ-function.

$$\int_0^\infty t^n e^{-t}\, dt = n! \quad (n = 0, 1, 2, \ldots)$$

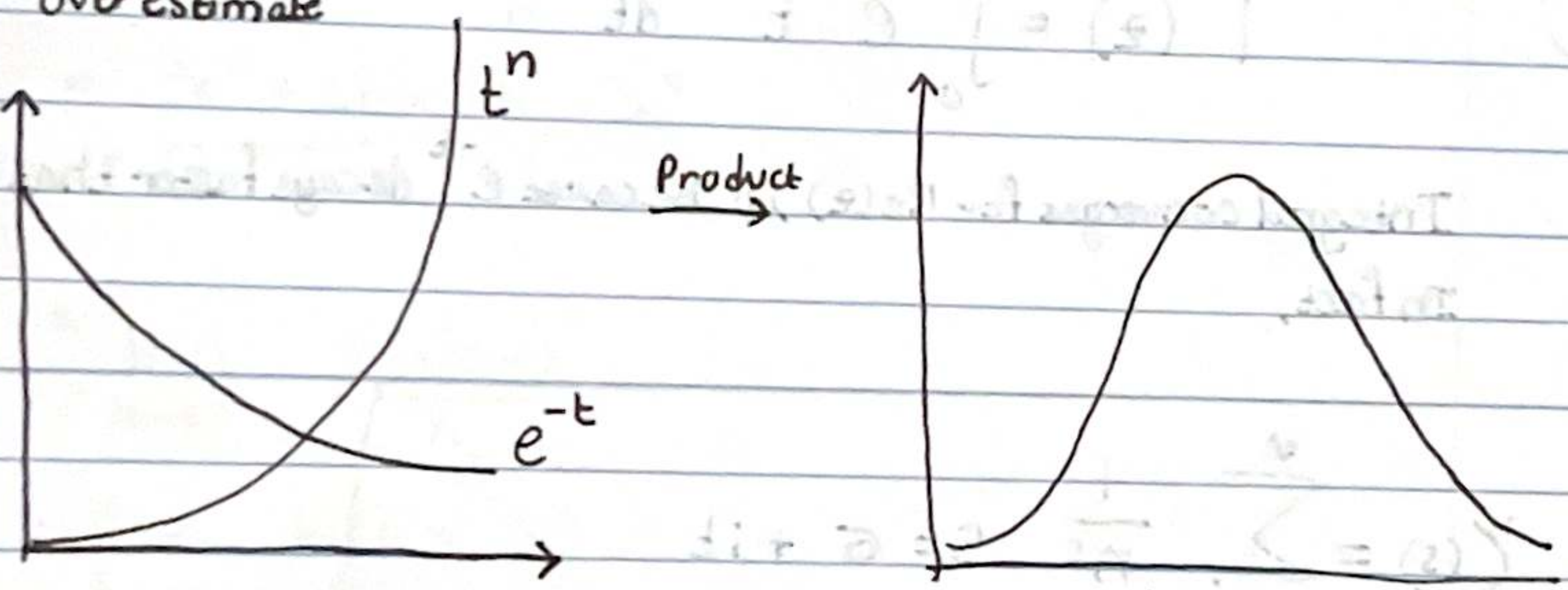
$$\int_0^\infty t^0 e^{-t}\, dt = 0! \equiv 1 \longrightarrow \text{Definition of } 0!$$

Sterling's Formula logarithmic correction.

$$n! = n^n e^{-n} \sqrt{2\pi n}$$

compensating factor

overestimate



Can be replaced by a gaussian integral

① Contribution is mind at extreme values.

$$\int_0^\infty t^n e^{-t}\, dt$$

$$\downarrow$$

$$\int_0^\infty e^{-(t - n\ln t)}$$

?

The Gamma (Γ) function.

It is a meromorphic function which extends the factorial function to all complex numbers other than non-positive integers.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$$

Integral converges for $Re(z) > 0$ because e^{-t} decays faster that t^{z-1} grows.

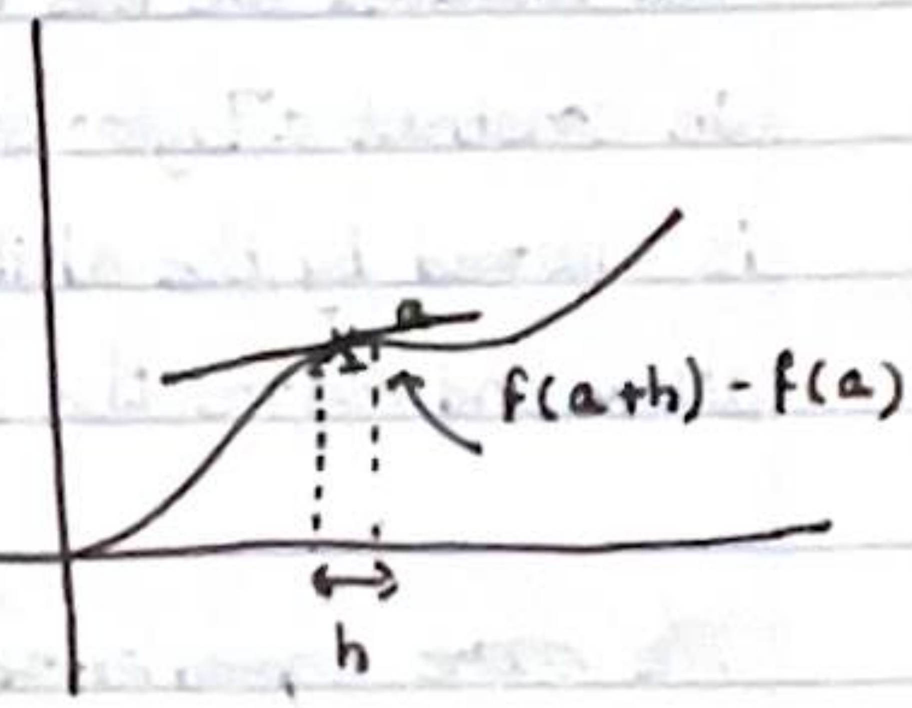
In fact,

$$\boxed{\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{n^s} \quad s = \sigma + it \quad \to Re(s) > 1 \text{ (Converges)}}$$

Differentiation

The first derivative of f at a,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



This gives us $f'(a)$ which are coefficients for our power Series

We notice that $\frac{d}{dx}(x-a)^n = n(x-a)^{n-1}$

$$\Rightarrow \frac{d^n}{dx^n}(x-a)^n = n!$$

To recover purely the contribution of the coefficients near a without differentiation multiplicative amplification, we exclude the factorial form $\Rightarrow \frac{1}{n!} \times f^n(a)$

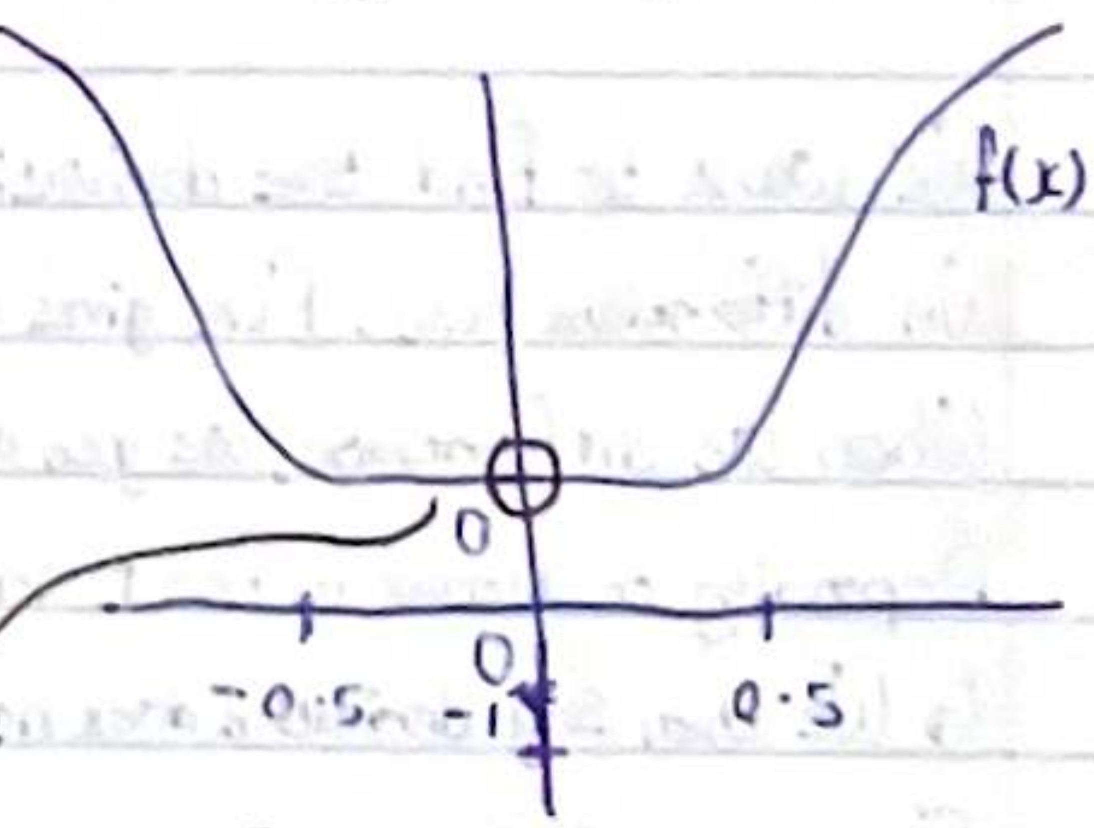
Now we have coefficients from taking derivatives.
~~Will the coefficients lead to the convergence of the series to~~ f ?

This function is continuous, smooth, but not analytic.

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Real Analysis & Complex Analysis.

Example, Smooth & non-Analytic function



Derivatives exist everywhere
But taylor Series only converges at 0
it fails to converge anywhere else.

Analytic Continuation is not guaranteed.

The thing about ~~real~~ analysis is that it sees a function piece by piece. In the real case, the function is modular. It might be smooth & Analytic on one interval and fail in the other. The Real case has no pattern to ensure the next piece will be well behaved. Complex analysis is more rigid. It is like a fractal. If it is well behaved for one piece, then it is well behaved on the whole domain. Analytic continuation is more like

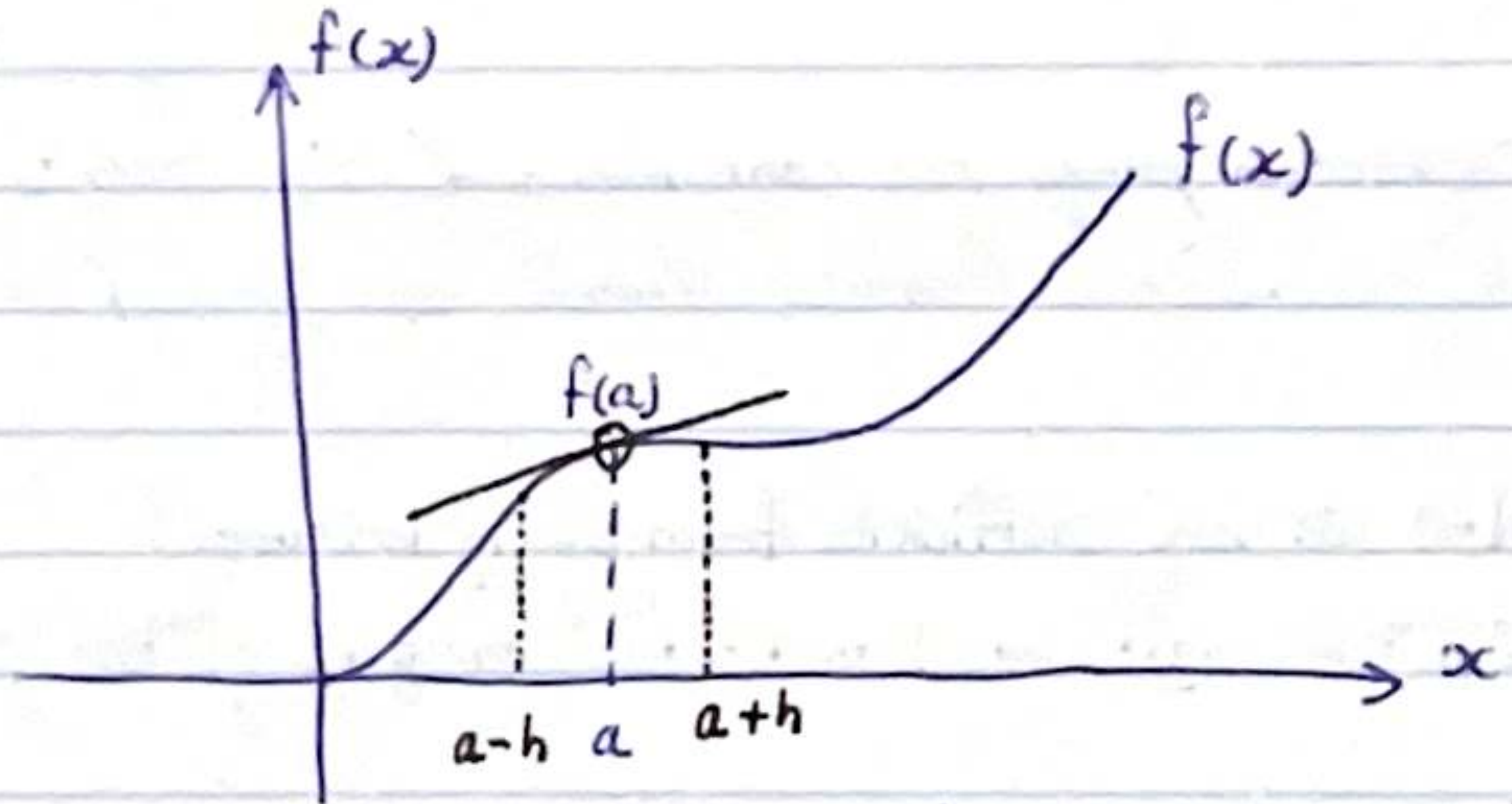
Real functions are less likely to be well behaved, or predictable. To assess their predictability, we construct a Taylor Series to approximate them. The "Wellness" of their behaviour is marked by the ability of their Taylor Series to converge. We try to think a bit ahead and require the series to converge for a neighbourhood instead of just a point.

The more unpredictable they are, the more likely ~~the~~ the Series will fail to converge.

Complex functions are more well behaved. They exhibit some kind of fractal order.

Our Test makes use of Taylor Series, hence Analyticity, hence Smoothness, hence differentiability.

Differentiation in $\mathbb{R}$



We want to find the derivative at a.

$f'(x)$ is a function encoding variation in the gradient of a line tangent to ~~a~~ point x.

We differentiate $f(x)$. $f'(x)$ gives us the gradient of a line tangent to the point in the argument.
When we differentiate, we get a general solution to the slope.
Depending on breaks in the function, Some of the result might be nonsense.
↳ We then Say derivative does not exist at that point.
$f'(x)$ gives derivative only at the points where $f(x)$ is differentiable.

$m_{tangent}$ at x

$$f'(x) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

How we were taught derivative.
But it must approach from both sides and limits must agree.

How despite these methods of ensuring derivative and smoothness, some function can be smooth and not analytic?

Differentiability & Smoothness in $\mathbb{R}$ seem to have nothing to do with convergence of Taylor Series.

Smoothness

Something apart from differentiation in $\mathbb{R}$ is causing Taylor Series to diverge. ??

~~That it~~

But in $\mathbb{C}$, the smoothness/differentiation allows convergence.

There is something different in their differentiation.

$$\frac{f^n(a)}{n!} \to 0$$ very fast in non-analytic function.

$f^n(a)$ goes to 0 faster than $n!$

$O(n!)$

Factorial Time

Solving travelling Salesman through Brute force means

Very bad

Grows very quickly.

lose information too fast.

Derivatives encode information

That information from first principle depends on neighbors ($a+h$)

Poor neighbors lead to unstable information & hence faster decay.

→ Loss of information / Quality of information for derivative in $\mathbb{R}$ is poorer than $\mathbb{C}$.

How does derivative from first principles generate information?

I need to show non-analytic function over-flow the information capacity of real derivative.

Complex derivative has a way of ensuring better neighbors hence more stability.

Is it real derivative failing to capture information or $\mathbb{R}$ has more freedom than $\mathbb{C}$?

I know complex differentiation forms a ball around the function compared to limits 2 sides in ~~$\mathbb{R}$~~ $\mathbb{R}$.

Still, even if we take h or the ball small, we can miss weird behavior.

Complex Differentiation follow ~~Cauchy~~ CR Equations which imply satisfying Laplace equation

→ Equilibrium.

in $f(z)$

equation

→ Real & Imaginary parts satisfy laplace ~~eq~~.

Equations / functions in $\mathbb{R}$, $f(x)$ do not need to satisfy laplace equation

Laplace Equations providing means of local stability to get good information?

Real Analytic Functions & Complex analytic functions

Functions of Real Variables can or cannot be analytic
This depends on the function.
For the Real Case, it has nothing to do with a function being Smooth (differentiability)

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & , x = 0 \end{cases}$$

function does not equal Taylor-Series for points $x \neq 0$

Smooth but not Analytic.

Complex ~~Smoothness~~ is more restrictive.

A non-continuous function is not differentiable at point of break thus cannot be the smooth on the interval.

$\boxed{\text{Conditions : Differentiable, Analytic, Continuous, Smoothness}}$

~~For the real case, we need a function that is~~
~~i) not smooth~~
~~ii) Smooth & Analytic~~
~~iii) Smooth & non-Analytic.~~

Suppose we have a function f and we are interested at point a.
The function needs to be continuous at that point. (No gaps)
The function must ~~be~~ have first order derivative [differentiable]
The function must be infinitely differentiable [Smooth]
The Taylor Series must converge to the function [Analytic]
A function can also be smooth & non-differentiable.

Smoothness does not imply analyticity in the real domain but it does in the complex one.
This hinges on differentiability in each.

Some how, despite being infinitely differentiable, f can escape analyticity in $\mathbb{R}$ but not in $\mathbb{C}$
Graphically, how does differentiation allow power series to escape in $\mathbb{R}$ but not in $\mathbb{C}$:

What is differentiation & how does power series relate to it?

Sequence $(a_n)_{n=0}^{\infty}$ → String of characters

Series $\sum_{n=0}^{\infty} a_n$ → Sum of a Sequence

Power Series $\sum_{n=0}^{\infty} a_n(x-c)^n$ → form a ring $R[[x]]$

Each term involves a variable in x.

Centre of the Series

Series can converge or diverge. Same for power Series.

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$ $\quad r<1 \Rightarrow$ Absolute Convergence.

Radius of convergence $R = 1/r$

Disk of Convergence

↳ $(c-R, c+R)$ where c is the Centre of the power Series

<u>Taylor Series</u>

↳ Power Series used to represent a function.

↳ Makes use of sum of derivatives of the function ~~around the centre of the ce~~ using the point of interest of centre of the series.

To be an infinite Series, we need infinite terms → Infinitely differentiable, $f \in C^{\infty}$

$a_n = \frac{f^n(c)}{n!}$ [Smoothness]

↳ What about Taylor Series of functions with finite Smoothness (differentiability).

↳ Get only linear or quadratic approximation for x^2 or x^3

→ Note: This infinite Series can Converge or Diverge.

Independent of Smoothness

The Series Can converge but not to its function.

We also desire the Taylor Series to Converge to the function f.

If there is such a function exists,

For a neighborhood c, at that point the Taylor Series converges to $f(c)$, and is valid in the neighborhood c, then f is an analytic function.

Not All functions have a taylor expansion

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(x_0)}{n!}(x-x_0)^n}$$

1) $f \in C^{\infty}$. The function must be infinitely differentiable to have a power series expansion. → Necessary for taylor series to exist.

2) The Series must Converge

3) For Taylor Series at x_0, the taylor Series must Converge to the function $f(x)$. It could be that the taylor Series converges to Something different from $f(x)$.

Only Analytic functions guarantee that the Taylor Series of $f(x)$ converges to $f(x)$.

[This is local at x_0]

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n \qquad a_n = \frac{f^n(x_0)}{n!}$$

$$\boxed{\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}}$$

Gives the radius of convergence of Taylor Series.

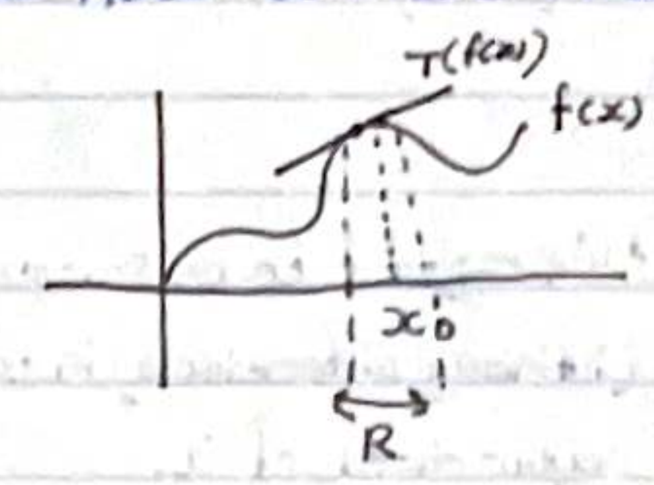
Cauchy Hadamard is in Complex Analysis.

also $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ if limit exist.

Real Analytic Functions

What We previously described is Real Analytic functions.
f is Said to be real analytic at x_0 if f is infinitely differentiable and ~~converges~~ its Taylor Series converges to $f(x)$ in neighbourhood of x_0 pointwise.

Notice how We are restricted to the neighbourhood of x_0.



We might want to extend this radius of convergence.

Analytic Continuation of Real Analytic function.

↳ We Want a taylor Series that holds beyond the Radius of Convergence.

Taylor Expansion of a function. → Invention of electronic Calculators undermine the usefulness of this.

I have functions like e^x. For e^2, I can do $(2.7)^2$ and get a weak approximation. But for things like $e^{0.2}$, $e^{-3.5}$, e^{ix}, this becomes a nightmare; and the approximation sucks too !!

I want to compute $\sqrt{1.01}$ (Difficult to do mentally)

Taylor expand $\sqrt{x}$ around $x_0 = 1$

$$\sqrt{x} = \sqrt{1+h} \approx 1 + \tfrac{1}{2}h - \tfrac{1}{8}h^2 + \dots$$

$h = 0.01$

$\Rightarrow \sqrt{1.01} = 1.005$

Convert this hard problem to simple polynomial powers

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x-x_0)^n}$$

~~For Power Series Expansion,~~

~~Taly~~

Taylor Series require the function $f(x)$ to be ANALYTIC at the point of defn x_0

TO REPRESENT THE FUNCTION

Infinitely Differentiable

$[f \in C^\infty] \rightarrow$ Smooth

The function CONVERGES to its taylor Series at x_0

NO

A function may differ from its taylor series sum.

When we looked at Taylor Series, we saw it as a useful means of approximating a function at a point.

The generation of the Taylor Polynomial involved coefficients $\frac{f^n(a)}{n!} = a_n$

$n!$ blows up really fast. ~~The series diverge when $f^n(a)$ goes to 0 faster than $n!$ blowing up.~~

↳ It ensures ~~that the functi~~

dampening of the Series for large values.

To ensure convergence of T.S to the true value of the function, the derivative term $f^n(a)$ is important.

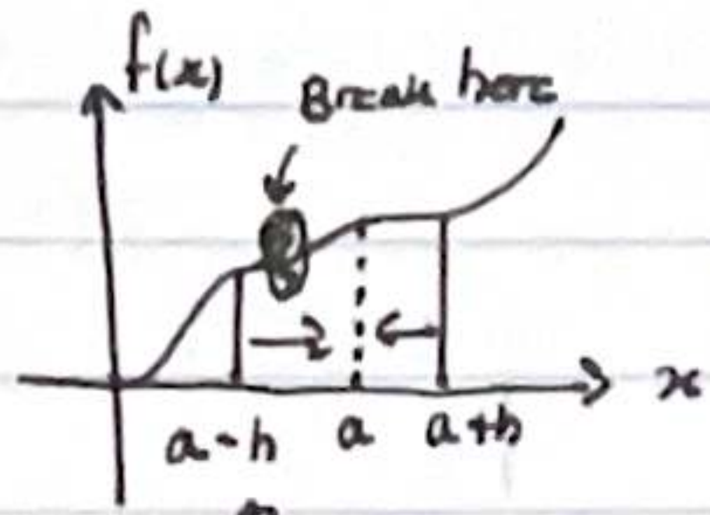
Analyticity (at a point a) is determined by the derivative $f^n(a)$.

Analyticity means the T.S converges to the function in the neighbourhood of that point.

That depends on the derivative.

Investigating the derivative will explain why some functions fail to be analytic.
We can also investigate the process of taking a derivative.

The derivative of that order must be sensitive enough to capture the details of the function.



Left side limit does not exist.

$$\boxed{\lim_{x\to a} f(x) \text{ exist if and only if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)}$$

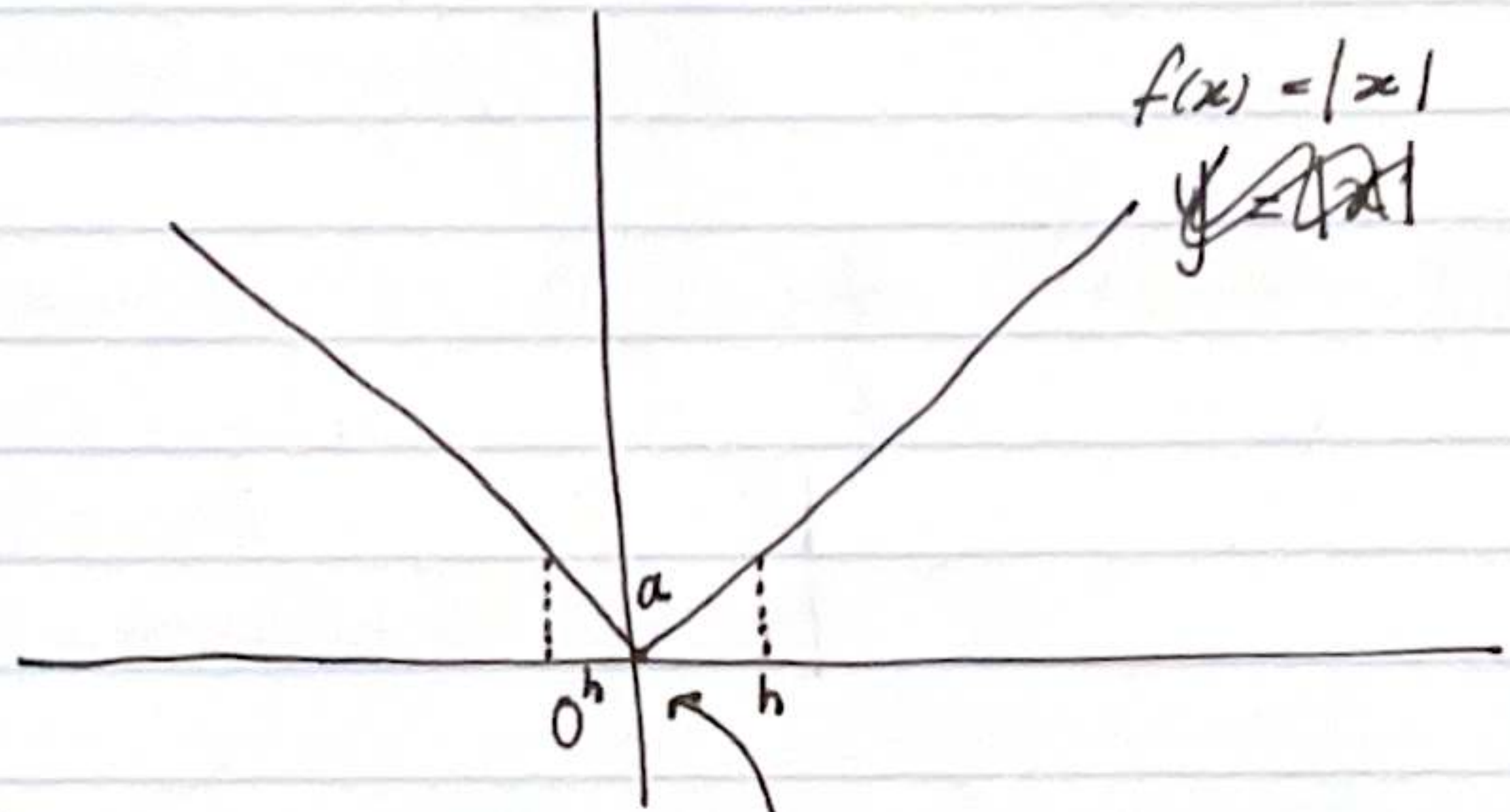
limit exists if both agree. (and they exist)

We might want to write

$$\frac{f(a-h)-f(a)}{h}$$

h must be >0 to approach from left

Review that again



There's a corner here

Left-hand limit

$\lim_{x\to a^-} f(x)$ x is smaller than a and is getting closer to a as we reduce the gap.

this gives th magnitude of h, +ve or −ve

Left-hand derivative

$$f'_-(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$$

Same formula.

Right-hand limit

$\lim_{x\to a^+} f(x)$ x is larger than a and gets closer.

Right-hand derivative

$$f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$

e.g. $f(x) = |x|$ $\quad f(x) = \begin{cases} -x, & x<0 \\ x, & x\geq 0 \end{cases}$

limit ONE for $f(0)$

~~$f'_-(x) =$~~

$$f'_-(0) = \frac{f(x+0)-f(0)}{h} \quad x<0 \Rightarrow h<0 \text{ for } \lim_{h\to 0^-} \quad = \frac{|h|-0}{h} = \frac{-h}{h} = -1$$

$$f'_+(0) = \frac{f(x+0)-f(0)}{h} \quad x>0 \Rightarrow h\geq 0 \text{ for } \lim_{h\to 0^+} \quad = \frac{|h|-0}{h} = \frac{h}{h} = +1$$