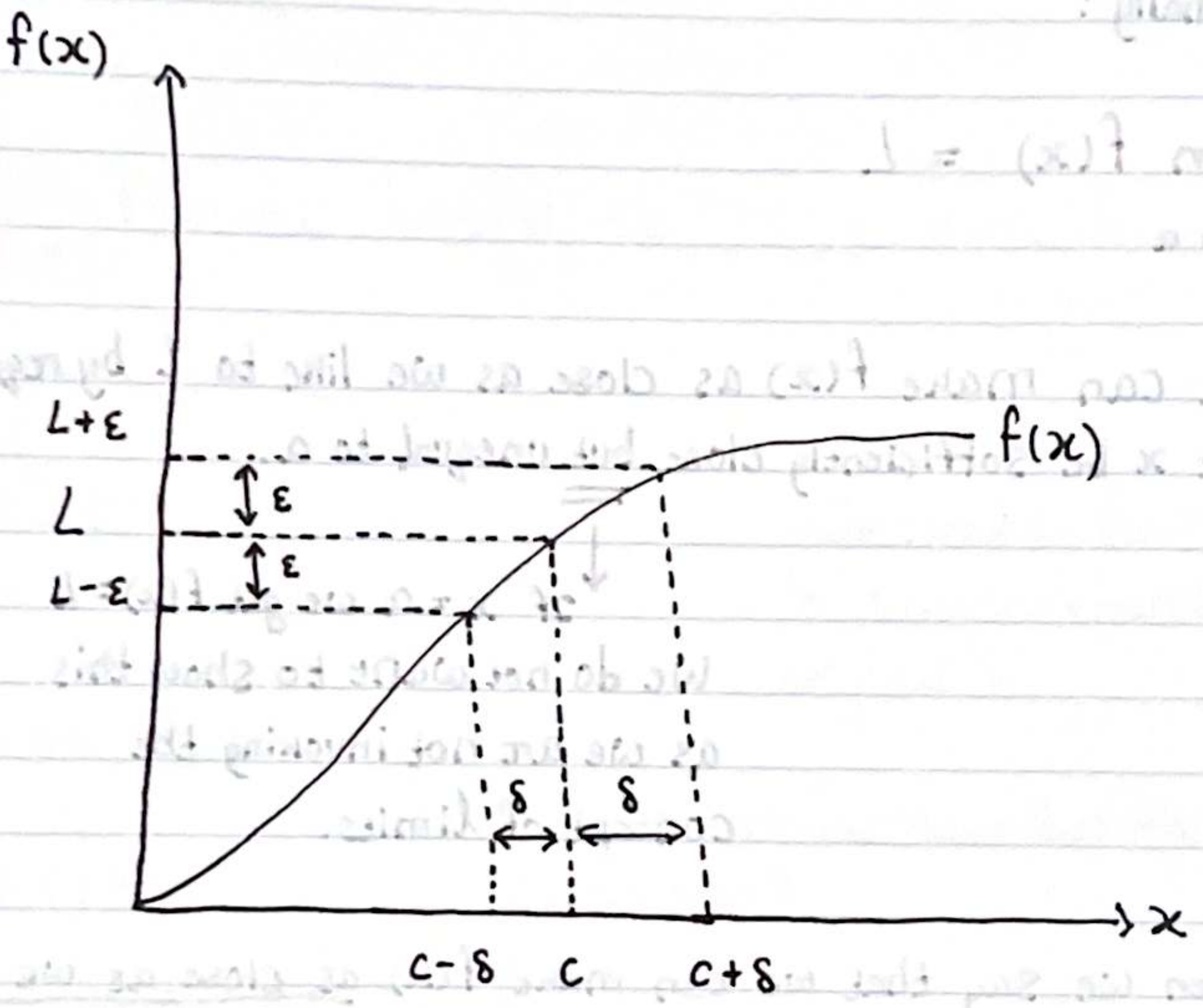
$(\varepsilon - \delta)$ defn of limits.



Given $\lim_{x \to c} f(x) = L$

Challenge : We must show that for every chosen (non-zero) ε, there always exists a corresponding δ.

How is this always condition satisfied??.

This indicates / demonstrate that for all infinitesimally small range of values ε close to L we can always find a range of value δ.

Value is not absolutely defined but is in a range.

This rigourously defines limits without invoking $\lim_{x \to c}$ operator.

In Formally:

$$\lim_{x \to a} f(x) = L$$

- We can make $f(x)$ as close as we like to L by requiring that x be sufficiently close but unequal to a....

  If $x = a$ we get $f(x) = L$

  We do not want to show this as we are not invoking the concept of limits.

- When we say that we can make $f(x)$ as close as we like to L we mean that for all non-zero distance ε, we can make the distance between $f(x)$ and L smaller than ε.

(ε, δ) defn

A function f approaches the limit L near a if we can make $f(x)$ as close as we like to L by requiring that x be sufficiently close to but unequal to a.

This means that For every non-zero distance ε, there is some non-zero distance δ such that if the distance between x and a is less than δ then the distance between $f(x)$ and L is smaller than ε. on x-axis

We need to find a value smaller than δ to show that a value smaller than ε exists on y-axis. Initial separation

Symbolically,

$$\lim_{x \to a} f(x) = L \iff \left(\forall_\varepsilon > 0, \exists \delta > 0, \ 0 < \underbrace{|x-a|}_{} < \delta \Rightarrow |f(x) - L| < \varepsilon\right)$$

We need to find a new δ smaller than the original δ. ??

But how do we show that this can go on forever?

It does not. We can be given an $\underbrace{\varepsilon \text{ infinitesimally}}_{}$ small and still find a new δ.

This is a proof as i) ε was not defined by us but by the challenger ii) who will try to ~~make~~ give us a very small ε (to try and oppose us) but if limit exists we will always find a δ.

NEGATION

$$\lim_{x \to c} f(x) \neq L$$

if $\exists_\varepsilon > 0$ s.t $\forall_\delta > 0$

$\exists x$ s.t $0 < |x - c| < \delta$

and $|f(x) - L| > \varepsilon$

Limits at ∞

$$\lim_{x \to \infty} f(x) = L$$

if $\forall_\varepsilon > 0$

$\exists N > 0$

s.t

$|x - 0| > N$

and which implies

$|f(x) - L| < \varepsilon$

Examples

1) Show

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

Assume: $\varepsilon > 0$ is given.
We need a δ (which is non-zero)
s.t

$$|x - 0| < \delta \Rightarrow \left| x \sin\left(\frac{1}{x}\right) - 0 \right| < \varepsilon$$

$\because \sin(x) \in [-1, 1]$

$$\left| \underbrace{x}_{\text{can be +ve}} \sin\left(\frac{1}{x}\right) - 0 \right| = |x| \left| \sin\left(\frac{1}{x}\right) \right|$$

1

can be +ve or -ve, corrected due to modulus

absolute

Else values will be $\in [0, 1] \times x$ ← with max the value $= x$

as -ve values become +ve.

$$\Rightarrow \boxed{\left| x \sin\left(\frac{1}{x}\right) - 0 \right| \leq |x|} \quad \text{— (1)}$$

We know that $|x| = |x - 0|$

We also know $|x - 0| < \delta$

~~Now if we let $\delta = \varepsilon$~~
~~$\Rightarrow |x - 0|$ and $< \varepsilon$~~

$\Rightarrow$ ~~$|x|$~~ $<$

Now, if we let $\delta = \varepsilon$
we get $|x| < \varepsilon$ (put in (1))

$$\Rightarrow \left| x \sin\left(\frac{1}{x}\right) - 0 \right| < \varepsilon \quad \text{as } |x| < \varepsilon \text{ itself}$$

$\Rightarrow$ Proof is complete.

Show

$$\lim_{x \to a} x^2 = a^2$$

Let $\varepsilon > 0$ be given

We need a $\delta > 0$ such that

$$|x-a| < \delta \implies |x^2 - a^2| < \varepsilon$$

$$|(x-a)(x+a)|$$

$$= |x-a||x+a|$$

By Triangle inequality

$|x+a| \leq |x| + |a|$ — ①

We recognise that $|x-a|$ is the term bounded by δ. We can pick a bound of 1 and later choose something smaller for δ.

$|x-a| < 1$ | $|x| - |y| \leq |x-y|$ generally holds for real numbers (Inverse $\triangle$ Inequality)

$$|x| - |a| < |x-a| < 1$$

$$|x| < 1 + |a| \quad \text{— ②}$$

② in ① yields

$$|x+a| < 2|a| + 1$$

We were given

$$|x-a||x+a| < \varepsilon$$

$$|x-a| < \frac{\varepsilon}{2|a|+1}$$

Now we set $\delta = \min\left(1, \frac{\varepsilon}{2|a|+1}\right)$

$\boxed{\text{So if } |x-a| < \delta}$

then $|x-a| < \frac{\varepsilon}{2|a|+1}$ ($\times\ |x+a|$ on both sides)

$$|x-a||x+a| < \frac{\varepsilon}{2|a|+1}|x+a|$$

$$|x-a||x+a| < \frac{\varepsilon}{2|a|+1}(2|a|+1)$$

$|x^2 - a^2| < \varepsilon \implies$ Proof is complete.

Prove

$$\lim_{x \to 5} (3x-3) = 12$$

Let $\varepsilon > 0$ be given.
We need a $\delta > 0$ Such that

$|x-5| < \delta \implies |3x-15| < \varepsilon$

$3|x-5| < \varepsilon$

We can let $\delta = \varepsilon/3$

$\implies |x-5| < \frac{\varepsilon}{3}$

We play with δ to get $|f(x)-L| < \varepsilon$

$\implies 3|x-5| < \varepsilon$

$|3x-15| < \varepsilon$

$\implies |3x-3-12| < \varepsilon$

$\implies$ Proof is Complete.

- Under examination:

  The key to the proof is the identification of the value of δ.
  To find that δ we begin with the final statement $|f(x)-L| < \varepsilon$ and work backwards until we reach the form $|x-c| < \delta$

Example1: Show
(linear)

$$\lim_{x \to 4} (5x-7) = 13$$

Given an $\varepsilon > 0$ we must find a $\delta > 0$ Such that

$|x-c| < \delta \implies |f(x)-L| < \varepsilon$

$|f(x)-L| < \varepsilon \rightarrow |5x-7-13| < \varepsilon$

$|5x-20| < \varepsilon$

$5|x-4| < \varepsilon$

Work out

$|x-4| < \frac{\varepsilon}{5}$

We can set $\delta = \varepsilon/5$

Proof 1

- Suppose $\varepsilon > 0$ has been provided.
- We need a $\delta > 0$ s.t $|x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

- Let $\delta = \frac{\varepsilon}{5}$

- Now $\forall x \quad 0 < |x-c| < \delta \Rightarrow |x-4| < \frac{\varepsilon}{5}$

$5|x-4| < \varepsilon$

$|5x-20| < \varepsilon$

$|(5x-7)-13| < \varepsilon$

$\Rightarrow$ Proof is complete.

Show $\lim\limits_{x \to a} x^2 = a^2$

$|x^2 - a^2| < \varepsilon$, $|x-a| < \delta$

$|x+a||x-a| < \varepsilon$

Via Δ Inequality

$|x+a| \leq |x| + |a|$

$\Rightarrow |x-a| < \frac{\varepsilon}{|x|+|a|}$ $\leftarrow$ We must remove x from R.H.S Argument for (ε, δ) proof to be valid.

X

As can be seen $|x-a|$ is the term bounded by δ

For a first instant, we can choose $\delta = 1$

$|x-a| < 1$

By inverse Δ Inequality,

$|x| - |a| < |x-a| < 1$

$\Rightarrow |x| < 1 + |a|$

$\Rightarrow |x-a| = \frac{\varepsilon}{1+2|a|}$ $\leftarrow$

Proof

Suppose $\varepsilon > 0$ is given.

We need a $\delta > 0$ s.t $|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Let $\delta = \min\left(1, \dfrac{\varepsilon}{1+2|a|}\right)$

Since 1 was chosen as "earlier bound")

Suppose $\delta = \dfrac{\varepsilon}{1+2|a|}$

$\Rightarrow |x - a| < \dfrac{\varepsilon}{1+2|a|}$

$|x - a||x + a| < \dfrac{\varepsilon}{\cancel{1+2|a|}} \cancel{|x+a|}$

$|x^2 - a^2| < \varepsilon \quad \Rightarrow \quad \lim\limits_{x \to a} x^2 = a^2$

$\Rightarrow$ Proof is complete.

Show $\lim_{x \to 5} (3x^2 - 1) = 74$

$|3x^2 - 75| < \varepsilon$ We need to get in form $|x - c| < \delta$

$-\varepsilon < 3x^2 - 75 < \varepsilon$

$-\varepsilon + 75 < 3x^2 < \varepsilon + 75$

$\cancel{75 - \varepsilon}$

$25 - \frac{\varepsilon}{3} < x^2 < 25 + \frac{\varepsilon}{3}$

$\sqrt{25 - \varepsilon/3} < x < \sqrt{25 + \varepsilon/3}$

$-5 + \sqrt{25 - \varepsilon/3} < x - 5 < -5 + \sqrt{25 + \varepsilon/3}$

Antisymmetric

$-\delta < x - c < \delta$

But we define $\delta > 0$

$\Rightarrow$ We need to use the large ε to define δ

ie. $-5 + \sqrt{25 + \varepsilon/3}$

$\Rightarrow \delta = \min\left(\underline{\underline{-5 + \sqrt{25 - \varepsilon/3}}},\ -5 + \sqrt{25 + \varepsilon/3}\right)$

corresponds to $-\delta$ but also leads to $\varepsilon > 75$

Show

$$\lim_{x \to 5} (3x^2 - 1) = 74$$

We have

$|x - 5| < \delta$

$|3x^2 - 1 - 74| < \varepsilon$

$|3x^2 - 75| < \varepsilon$

$|x^2 - 25| < \varepsilon/3$

We must get a bound of $x - 5$

$|x + 5||x - 5| < \varepsilon/3 \Rightarrow |x - 5| < \dfrac{\varepsilon}{3|x+5|}$

By Δ Inequality

$|x+5| < |x| + |5|$

$|x+5| < |x| + 5$

By inverse Δ Inequality

$|x| - |5| < |x - 5|$

$\downarrow$

We recognise $|x-5|$ is the term bounded by δ

Let us pick a bound of 1 and later choose something smaller.

$\cancel{|x| - 5 < |x - 5| < 1}$

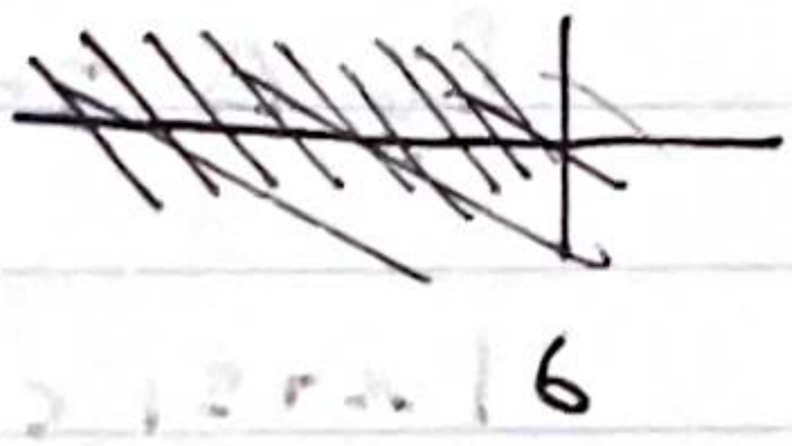
$\cancel{\Rightarrow |x| < 6}$

$|x - 5| < \dfrac{\varepsilon}{3|x+5|}$

By inverse Δ Inequality

$|x| - |5| < |x - 5| < 1$

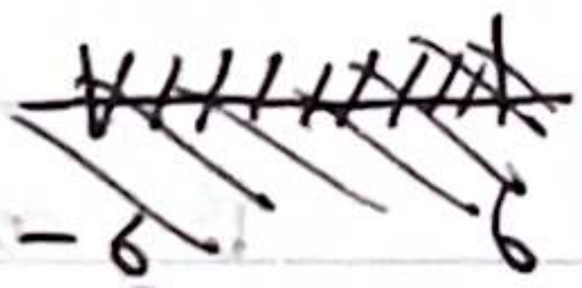
$\Rightarrow |x| < 6$



$|x+5| < |x| + 5$

$|x+5| < 11$

$\Rightarrow |x-5| < \dfrac{\varepsilon}{33}$??



$$|x+5||x-5| < \frac{\varepsilon}{3}$$

We get $|x-5| < \dfrac{\varepsilon}{3|x+5|}$

By Δ Inequality in $\mathbb{R}^1$

$$|x+5| < |x| + \underset{5}{|5|}$$

We recognise $|x-5|$ is the term bounded by δ
Let $\delta = 1$

By reverse Δ inequality
$$|x| - 5 < |x-5|$$

$$|x-5| < 1$$
$$\Rightarrow |x| < 6$$

$$\Rightarrow |x+5| = 11$$

$$|x-5| < \frac{\varepsilon}{33}$$

$$\delta = \min\left(1, \varepsilon/_{33}\right)$$
Suppose $\delta = \varepsilon/_{33}$

$$\Rightarrow |x-5| < \varepsilon/_{33}$$

let $\varepsilon_2 = \min(\varepsilon, K)$

Since $\varepsilon_2 > 0$ $\delta > 0$

When we say: $f(x)$ is close to L we mean $|f(x) - L|$ is small.

x and a are close $\rightarrow |x - a|$ is small.

When we say that we can make $f(x)$ as close as we like to L by requiring that x be sufficiently close to, but unequal to, a, we mean that for every non-zero distance ε, there is some non-zero distance δ such that if the distance between x and a is less than δ, then the distance between $f(x)$ and L is smaller than ε.

Proof method.

If one is provided with any challenge $\varepsilon > 0$ for a given f, a and L, one must answer with a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$. If an answer is provided, one has proven that the limit exists.

Definition

f(x) returns only real numbers i.e. $f(x) \in \mathbb{R}$.

→ range which f governs.

Let f be a real valued function defined on a Subset D of the real numbers. Let c be a limit point of D and let L be a real number.

↳ limit exists.

→ $\lim_{x \to c} f(x) = L$

If for every $\varepsilon > 0$ there exists a δ such that; for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

$$\lim_{x \to c} f(x) = L \iff (\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in D, \; 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$$

If $D = [a, b]$ or $D = \mathbb{R}$, then the condition that c is a limit point is automatically met because closed real intervals and the entire real line are perfect sets.

Note

$x \in [a, b] \rightarrow a \leq x \leq b \rightarrow$ closed interval

$x \in (a, b) \rightarrow a < x < b \rightarrow$ open interval

$x \in [a, b) \rightarrow a \leq x < b \rightarrow$ half-closed interval / half-open interval.

$x \in (a, b] \rightarrow a < x \leq b$

$x \in [a, \infty) \rightarrow a \leq x \rightarrow x \geq a \rightarrow$

$x \in (-\infty, a] \rightarrow x \leq a$

$x \in (a, \infty) \rightarrow a < x \rightarrow x > a$

eg: $x \leq 3$ or $x > 6$

$(-\infty, 3] \cup (6, \infty)$

$x \in (-\infty, a) \rightarrow x < a$

$(-\infty, 3] \vee (6, \infty)$

↑ or

Metric

$\vee$: or

$\wedge$: and

A metric (distance function) is a function that defines a distance between each pair of elements of a set. A set with a metric is called a metric space.

A metric on a set X is a function $d : X \times X \rightarrow [0, \infty)$

↓

Set of non -ve

$\forall x, y, z \in \mathbb{R}$

distance always exists between 2 real numbers

← Separation axiom.

① $d(x, y) \geq 0 \longrightarrow$ points (unless $x = y$) (non -ve)

② $d(x, y) = 0 \iff x = y \rightarrow$ distance between same point $= 0$

③ $d(x, y) = d(y, x) \rightarrow$ distance between x and y = distance between y and x.

④ $d(x, z) \leq d(x, y) + d(y, z) \rightarrow$ Triangle inequality.

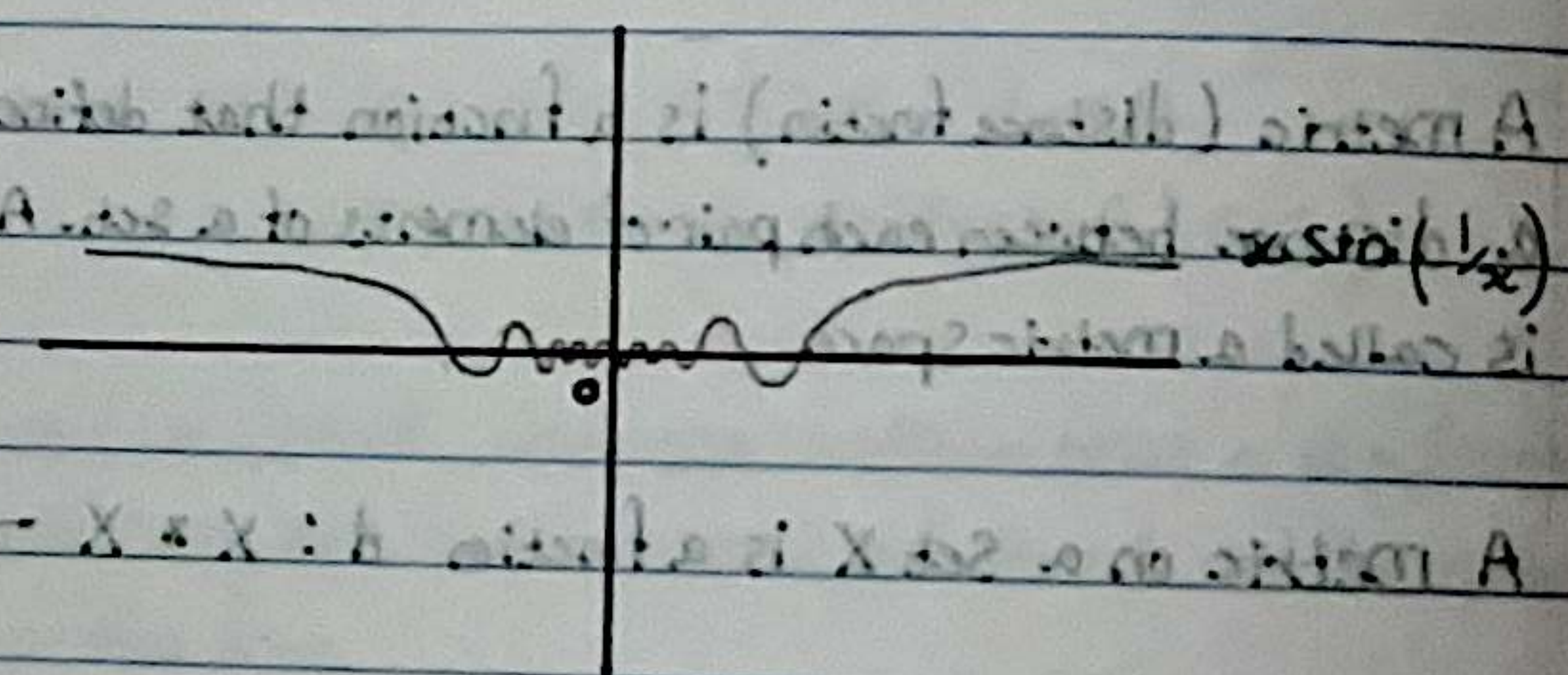
Elementary Combinatorics

Permutation and Combinations

Show that

① $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$

We let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that $|x - 0| < \delta$ implies $\left| x \sin\left(\frac{1}{x}\right) - 0 \right| < \varepsilon$



Since sine is bounded above by 1 and below by -1,

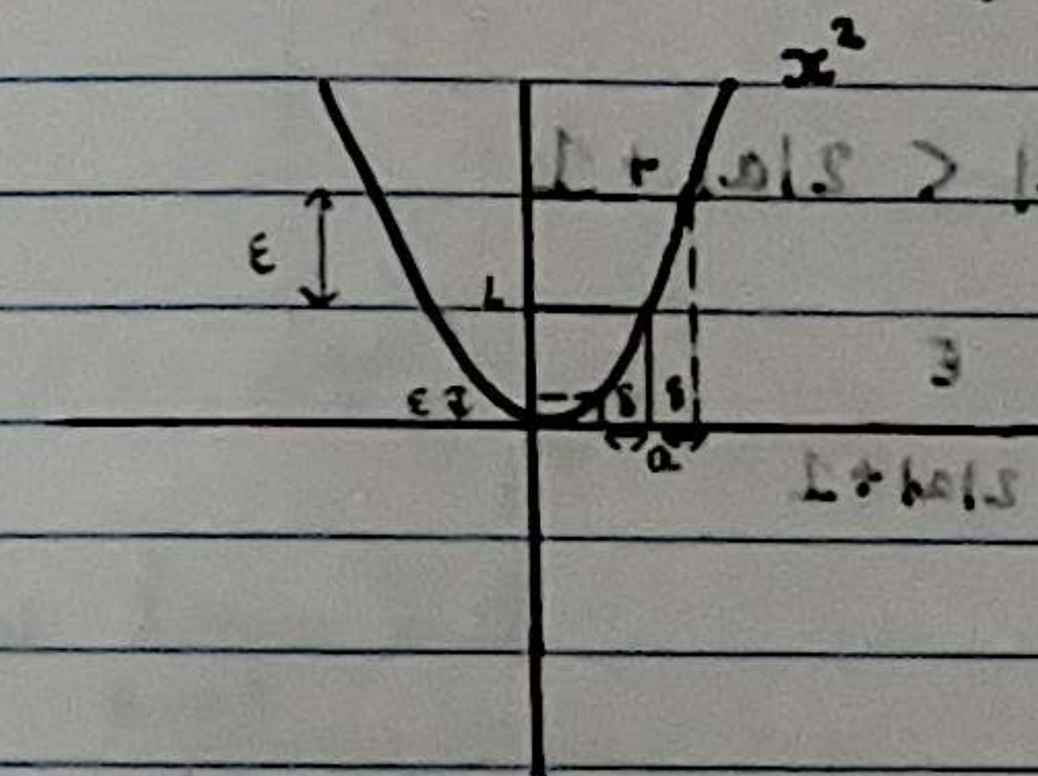
$$\left| x \sin\left(\tfrac{1}{x}\right) - 0 \right| = \left| x \sin\left(\tfrac{1}{x}\right) \right| = \underbrace{|x|}_{\leq |x|} \underbrace{\left| \sin\left(\tfrac{1}{x}\right) \right|}_{1}$$

Thus if we take $\delta = \varepsilon$, then $|x| = |x - 0| < \delta$ implies:

$$\left| x \sin(1/x) - 0 \right| \leq |x| \leq \varepsilon$$

Which completes the proof.

② Show that : $\lim_{x \to a} x^2 = a^2$ for any $a \in \mathbb{R}$



Let $\varepsilon > 0$ be given. We will find a $\delta > 0$ such that $|x - a| < \delta$ implies $|x^2 - a^2| < \varepsilon$

① $|x^2 - a^2| = |(x-a)(x+a)| = \underbrace{|x-a|}_{\text{bounded by } \delta\ (\to 0 < |x-a| < \delta)} |x+a|$

Suppose a bound of 1 and later pick something smaller for δ.

Suppose $|x-a| < 1$.

Since $|x|-|y| \leq |x-y|$ holds in general for real numbers x and y,

we have $|x|-|a| \leq |x-a| < 1$

Thus, $|x| < 1+|a|$

Via triangle inequality,

$|x+a| \leq |x|+|a| < 2|a|+1$

Suppose $|x-a| < \frac{\varepsilon}{2|a|+1}$

then $|x^2-a^2| < \varepsilon$

In Summary, we set $\delta = \min\left(1, \frac{\varepsilon}{2|a|+1}\right)$

So, if $|x-a| < \delta$, then

$|x^2-a^2| = |x-a||x+a|$

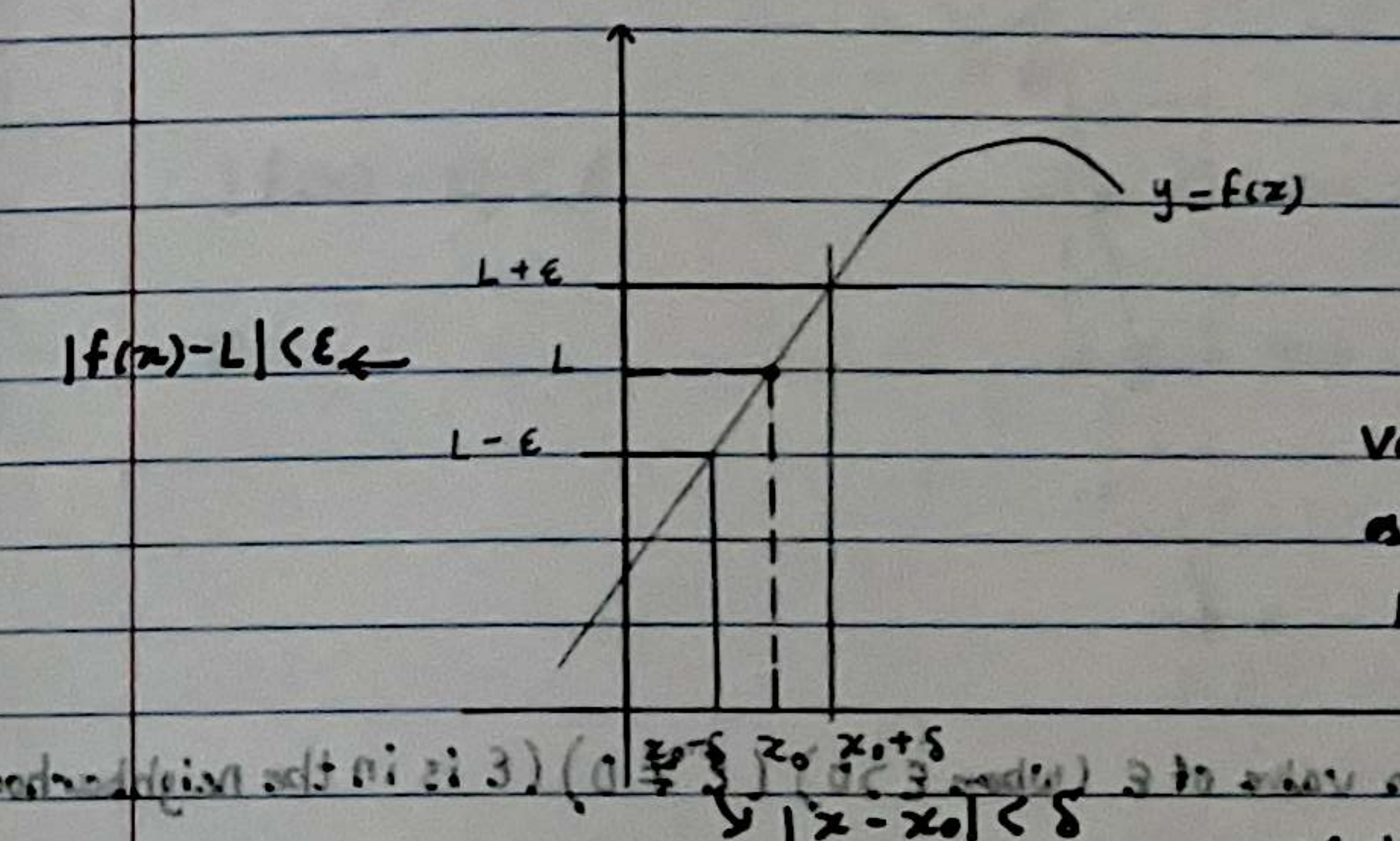
$< \frac{\varepsilon}{2|a|+1}(|x+a|)$

$< \frac{\varepsilon}{2|a|+1}(2|a|+1)$

$= \varepsilon$

Thus, we have found a δ such that $|x-a| < \delta$ implies $|x^2-a^2| < \varepsilon$

$\therefore$ we have shown that $\lim_{x \to a} x^2 = a^2$.



Given a value of ε, after knowing this value, we can determine a corresponding value for δ.

Value of δ is often given as a function of ε.

NOTE: multiple values of δ can be given.

If there is any value of ε for which we cannot find a corresponding value for δ, then the limit does not exist.

Show that $\lim_{x \to \pi} x = \pi$

Let $f(x) = x$

We need to determine what value our δ is going to have.

When $|x-\pi| < \delta$ we want $|f(x)-\pi| < \varepsilon$

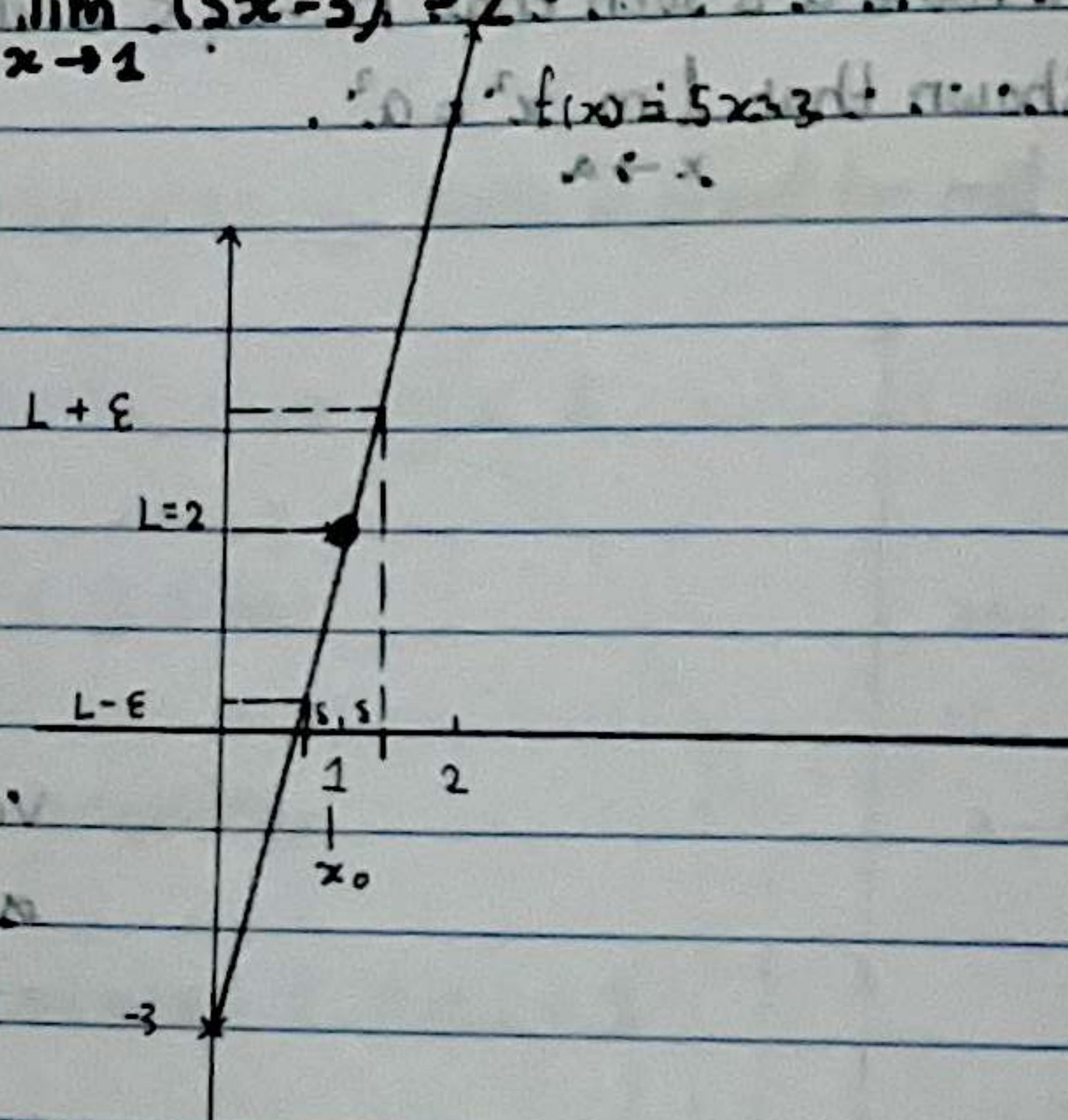
We know that $|f(x)-\pi| = |x-\pi| < \delta$ when so we take $\delta = \varepsilon$, we will have the desired property.

We could also have chosen $\delta = \varepsilon/2$

If $|x-\pi| < \delta = \varepsilon/2$, then

$|f(x)-\pi| < \varepsilon/2 < \varepsilon$ as required.

Show that $\lim_{x \to 1} (5x-3) = 2$



For a given value of ε (when ε>0) (ε ≠ 0) (ε is in the neighborhood of L)

We need a δ (δ>0) (Non-zero) (Very small)
Such that if x is within distance δ of x_0
ie. $|x - x_0| < \delta \rightarrow |x-1| < \delta$ -- ①

then $f(x)$ is within distance ε of L
ie. $|f(x)-2| < \varepsilon$

$|5x-3-2| < \varepsilon$ ∴ So, $\delta = \frac{\varepsilon}{5}$
$|5x-5| < \varepsilon$
$5|x-1| < \frac{\varepsilon}{5}$ -- ②

Verify

$|5x-3-2| = 5|x-1| < 5\left(\frac{\varepsilon}{5}\right)$ $= \varepsilon$

$\therefore |f(x)-2| < \varepsilon$

■

Prove that $\lim_{x \to 7} (x^2+1) = 50$

$L = 50$

$f(x) = x^2 + 1$

For a given value of $\varepsilon > 0$,

We need a δ such that $|x-7| < \delta$

$\therefore |x^2 + 1 - 50| < \varepsilon$

$|x^2 - 49| < \cancel{5}\ \varepsilon$

$|x+7|\,|x-7| < \varepsilon$

$\delta = \dfrac{\varepsilon}{|x+7|}$

$\varepsilon = \delta\,|x+7|$

$\delta > 0$

$|x-7| < \delta$

Assume

$|x-7| < 1$

$|x| < 8$

$|x+7| < |x| + |7| = 15$ (Triangle inequality)

$\delta = \dfrac{\varepsilon}{15}$

let $\delta = \min\left(1, \varepsilon/15\right)$

$|x^2 + 1 - 50| < \delta\,|x+7|$

$< 15\delta$

$< \varepsilon.$

∎

$\therefore \cancel{|x^2 - a^2}$

$|x - a| < \dfrac{\varepsilon}{2|a|+1} \quad \times |x + a|$

$|x - a||x + a| < \dfrac{\varepsilon}{2|a|+1} \times (2|a|+1)$

$< \varepsilon$

$\blacksquare$

Show that : $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$

Let $\varepsilon > 0$ be given. We need a $\delta > 0$: $|x - 0| < \delta \rightarrow \left|x \sin\left(\frac{1}{x}\right) - 0\right| < \varepsilon$

$|x| < \delta$

$|x| \, \underbrace{\cancel{\left|\sin(1/x)\right|}}_{|\text{max/min value}| = 1} < \varepsilon$

$|x| \cdot 1 < \varepsilon$

$|x| < \varepsilon$

$\therefore \varepsilon = \delta$

$|x| < \varepsilon$

$|x \sin(1/x)| < \varepsilon$

$\blacksquare$

Show that $\lim_{x \to 7} x^2 + 1 = 50$

Let a $\varepsilon > 0$ be given.

We need a $\delta > 0$ Such that $|x-7| < \delta$ implies $|x^2+1-50| < \varepsilon$

from $|x^2+1-50| < \varepsilon$

$|x^2 \cancel{+} - 49| < \varepsilon$

$|x+7|\,|x-7| < \varepsilon$

Let $\delta = 1$

$|x-7| < \delta$

$|x-7| < 1$

$|x| - |7| < 1$

$|x| < 8$

Now, $|x-7| < \delta \frac{\varepsilon}{|x+7|}$ → Via Δ inequality $|x| + |y| \leqslant |x+y|$

$|x-7| < \frac{\varepsilon}{15}$

$\delta = \min\left(1, \frac{\varepsilon}{15}\right)$

$|x-7| < \frac{\varepsilon}{15}$ ($\times |x+7|$ on both sides)

$|x-7|\,|x+7| < \frac{\varepsilon}{15} \underset{}{|x+7|}^{\nearrow 15}$

$|x^2+1-49| < \varepsilon$

∎

Using the $\varepsilon-\delta$ definition of a limit show that $\lim_{x \to 5} 2x+1 = 11$.

Given $\varepsilon > 0$, we need a $\delta > 0$ such that $|x-5| < \delta$ implies $|2x+1-11| < \varepsilon$

$|2x+1-11| < \varepsilon$
$|2x-10| < \varepsilon$
$2|x-5| < \varepsilon$
$|x-5| < \underbrace{\frac{\varepsilon}{2}}_{1^{st} \text{ value for } \delta}$

Working with ε

Also, let $\cancel{\delta = 1}$

$\cancel{|x-5| < 1}$
$\cancel{|x| - |5| < 1}$
$\cancel{|x| < 6}$

$|x-5| < \frac{\varepsilon}{2}$] put the ε in δ.

$2|x-5| < \varepsilon$
$|2x-10| < \varepsilon$
$|2x+1-11| < \varepsilon$

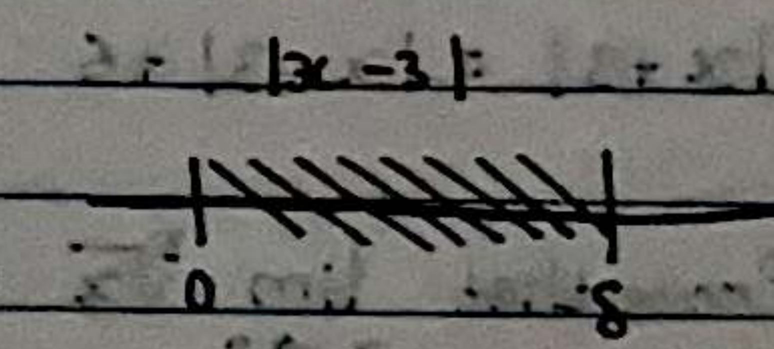
■

Using the $\varepsilon - \delta$ definition of limits show that $\lim_{x \to 3} x^2 = 9$.

① Given $\varepsilon > 0$, we need a δ such that ① $|x-3| < \delta$
② $|x^2 - 9| < \varepsilon$

From ②
$|x^2 - 9| < \varepsilon$
$|(x+3)(x-3)| < \varepsilon$
$|x-3| < \frac{\varepsilon}{|x+3|}$ — 1^{st} δ

Finding $|x+3|$
① we know that
$\delta > 0 \wedge \cancel{|x+3|}\ |x-3| < \delta$



$\therefore$ ~~0 < |x+3|~~
$\cancel{0 < |x-3| < \delta}$ $\cancel{(+6)}$
$\cancel{6 < |x+3| < \delta + 6}$

$|x-3| < \delta$
$|x+3| < \delta + 6$
Assuming $\delta < 1$

$|x+3| < \delta$

$|x-3| < \delta$

$$\lim_{x \to 3} x^2 = 9$$

$|x-3| < \delta \quad |x^2-9| < \varepsilon$

Given $\varepsilon > 0$

$|x+3||x-3| < \varepsilon$

$|x+3| = |x-3| + 6$

Prove that $\lim_{x \to 8} \sqrt[3]{x} = 2$

Given $\varepsilon > 0$, we need a $\delta > 0$ such that

$0 < |x-8| < \delta \implies |\sqrt[3]{x} - 2| < \varepsilon$

$\varepsilon - \delta$ definition of a limit

$-\varepsilon < \sqrt[3]{x} - 2 < \varepsilon$

$2 - \varepsilon < \sqrt[3]{x} < 2 + \varepsilon$

$(2-\varepsilon)^3 < x < (2+\varepsilon)^3$

$(2-\varepsilon)^3 < 8 - \delta < (2+\varepsilon)^3$

$-8 + (2-\varepsilon)^3 < -\delta < (2+\varepsilon)^3 - 8$

From $\dfrac{x_2^2 + B^2}{x_2 B + 1} = k > 0$

it futher follows that x_2 is a positive Integer.

Finally $A \geq B$ implies $x_2 = \dfrac{B^2 - k}{A} < A$ and thus $x_2 + B < A + B$ which

Contradicts the minimality of A and B.