Intermediate Value Theorem (IVT)

It states that if a continuous function f, with an interval $[a,b]$, as its domain, takes value $f(a)$ and $f(b)$ at each end of the interval, then it also takes any value between $f(a)$ and $f(b)$ at some point within the interval.

Corollaries

① If a continuous function has values of opposite sign inside an interval, then it has a root in that interval. - Bolzano's theorem.

② The image of a continuous function over an interval is itself an interval.

Theorem

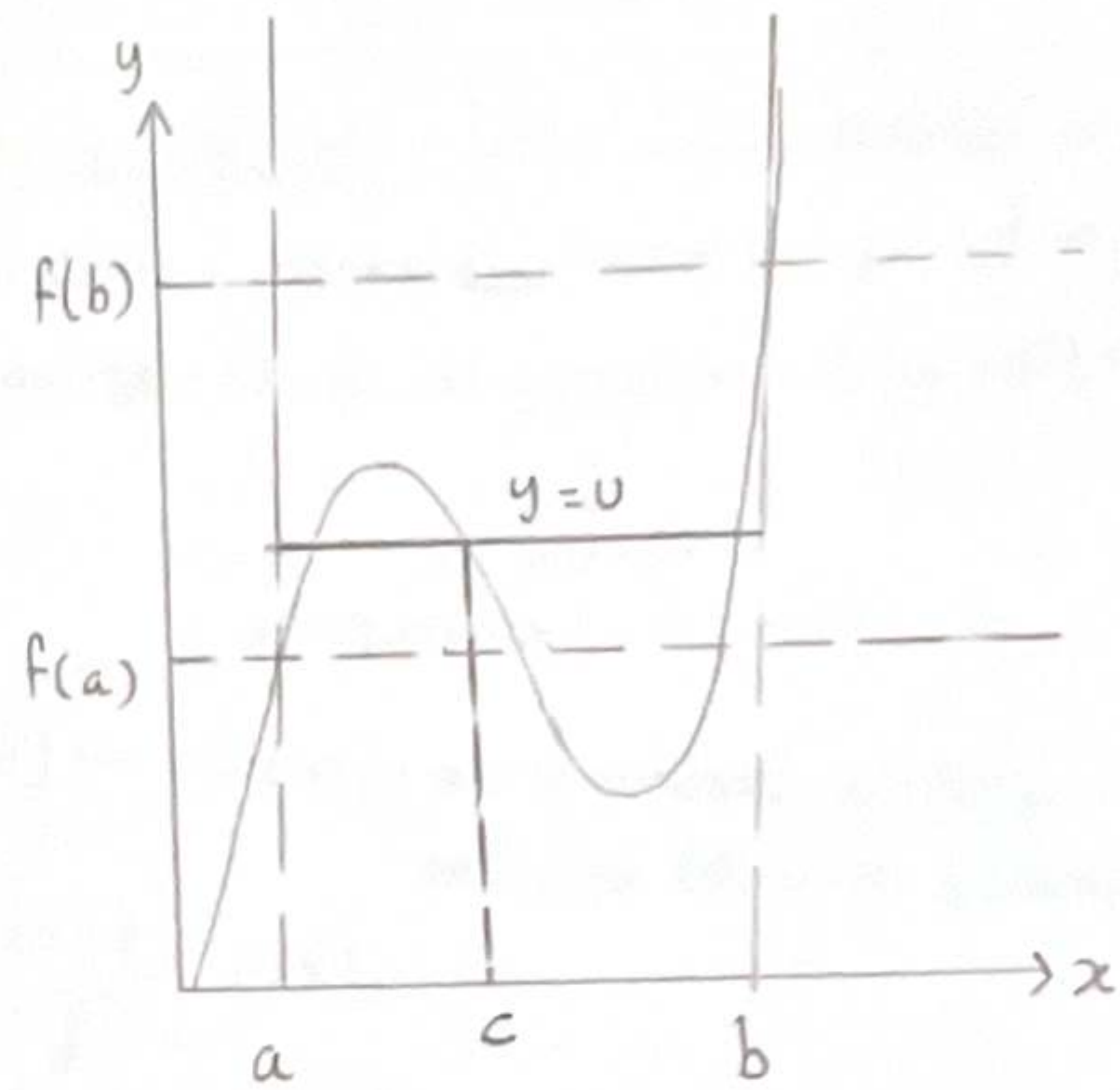
The intermediate value theorem states the following:

Consider an interval $I = [a,b]$ in the real numbers $\mathbb{R}$ and a continuous function $f : I \rightarrow \mathbb{R}$. Then,

if u is a number between $f(a)$ and $f(b)$

that is, $\min(f(a), f(b)) < u < \max(f(a), f(b))$

then, there is a $c \in (a,b)$ such that $f(c) = u$



Relation to Completeness

This theorem depends on and is equivalent to, the completeness of real numbers. The intermediate value theorem does not apply to rational numbers $\mathbb{Q}$ because gaps exist between rational numbers; irrational numbers fill those gaps. eg. $f(x) = x^2 - 2$ for $x \in \mathbb{Q}$ satisfies $f(0) = -2$ and $f(2) = 2$. However, there is no rational number x such that $f(x) = 0$ as $\sqrt{2}$ is irrational.

Proof

This theorem may be proven as a consequence of the completeness property of real numbers:

$f(a) < u < f(b)$ ①

Let S be the set of all $x \in [a,b]$ such that $f(x) \leq u$. Then S is non-empty since a is an element of S, and S is bounded above by b. Hence, by completeness, the supremum $c = \sup S$ exists. That is, c is the lowest number that is greater or equal to every member of S, $\Rightarrow f(c) = u$.

② Fix some $\varepsilon > 0$. Since f is continuous, there is a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$. This means that $f(x) - \varepsilon < f(c) < f(x) + \varepsilon$

for all $x \in (c - \delta, c + \delta)$. By the properties of the supremum, there exists $a^* \in (c - \delta, c]$ that is contained in S so for that $a^* \rightarrow f(c) < f(a^*) + \varepsilon \leq u + \varepsilon$

Choose $a^{**} \in [c, c+\delta)$ that will obviously not be contained in S, so we have $f(c) > f(a^{**}) - \varepsilon \geq u - \varepsilon$

Both inequalities $u - \varepsilon < f(c) < u + \varepsilon$ are valid for all $\varepsilon > 0$, from which we deduce $f(c) = u$ as the only possible value.

A Darboux function is a real-valued function f that has the intermediate value property. ie. for any two values a and b in the domain of f and any y between f(a) and f(b) there is some c between a and b with f(c) = y. The intermediate value theorem states that every continuous function. However, not every Darboux function is continuous ie. the converse of IVT is false.

eg : $f : [0,\infty) \rightarrow [-1, 1]$ defined by $f(x) = \sin\left(\frac{1}{x}\right)$ for $x > 0$ and $f(0) = 0$. This function is not continuous at $x = 0$ because the limit of f(x) as x tends to 0 does not exist, yet the function has the intermediate value property.

Darboux theorem states that all functions that results from the differentiation of some other function on some interval has the intermediate value property (even though they need not be continuous).

Mean Value Theorem.

Closed segment of a differentiable curve ($\widehat{AB}$)

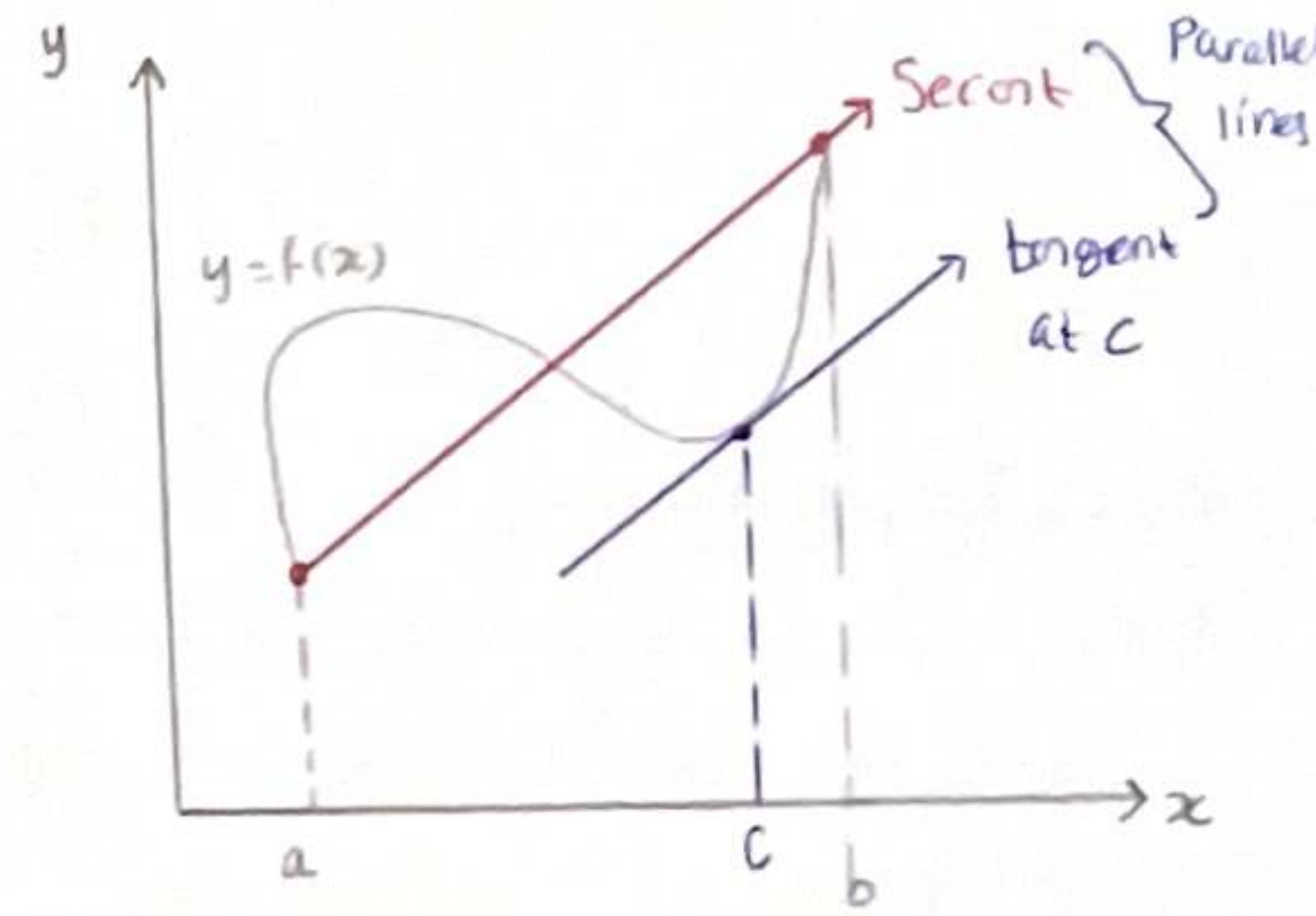
It states that, for a given planar arc between 2 endpoints, there is at least one point at which the tangent to the arc is parallel to the secant through its endpoints.

tangent → touches curve at 1 point.

secant → line that intersects a curve at 2 distinct points.

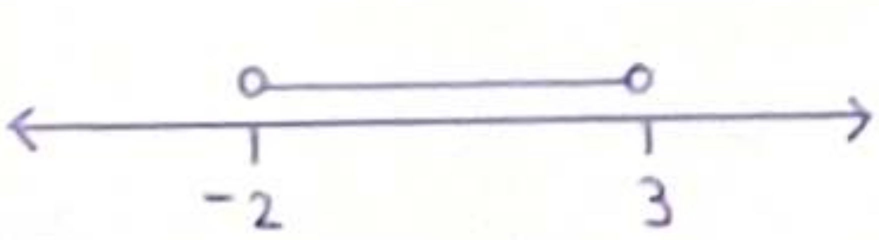
If f is a continuous function on the closed interval [a,b] and differentiable on the open interval (a,b), then, there exists a point c in (a,b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



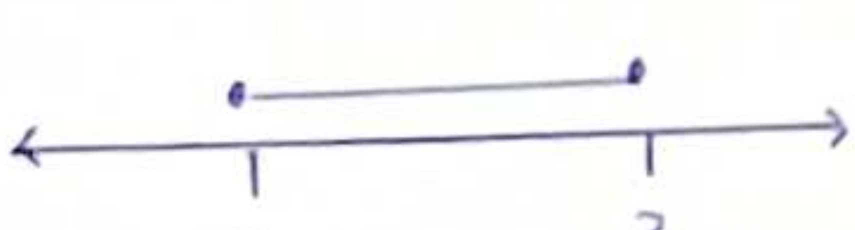
Interval Notation

① Open interval - Contains no endpoints



open interval (-2,3)

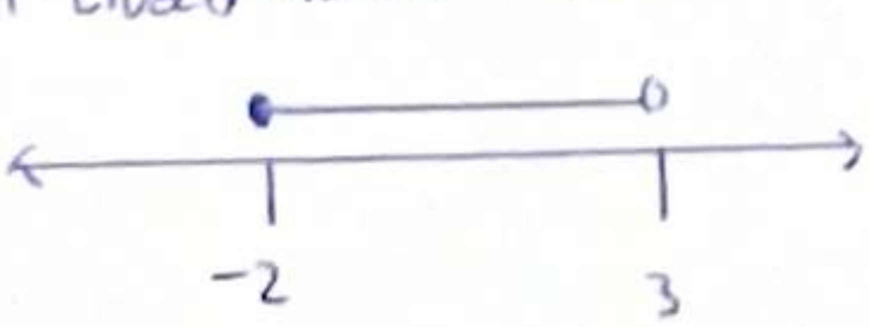
② Closed interval - Contains end points



closed interval [-2,3]

③ Half-closed interval - Contains 1 endpoint but not the other (or half-open)

(x,y] can also be ↑ & if opposite sides



half closed interval [-2,3)

④ Inclusive : Including the endpoints of an interval
eg Interval from 1 to 5 inclusive
⇒ Closed interval [1,5] = 1,2,3,4,5

⑤ exclusive : excluding end points
eg. interval from 1 to 5 exclusive
⇒ open interval (1,5) = 2,3,4
or]1,5[

Half-closed [1,5) = 1,2,3,4
or (1,5] = 2,3,4,5

eg. Solve $xe^x = 1$

Solution : $xe^x - 1 = 0$

Let $f(x) = xe^x - 1$

Let $f(x)$ be differentiable at any point and also (continuous)

$f(0) = 0 \cdot e^0 - 1 = -1 < 0$

$f(1) = 1e^1 - 1 = 1.718 > 0$

Change of sign indicates Presence of root. (Intersection with x-axis)

by Intermediate Value theorem, IVT → Proof.

f has a zero $(0, 1)$ → Interval

$f'(x) = xe^x + e^x.1$ → Product rule.

Let $x_0 = 0$

$$x_1 \approx 0 - \frac{0e^0 - 1}{0e^0 + e^0} = 1$$

$$x_2 \approx 1 - \frac{1e^1 - 1}{1e^1 + e^1.1} = 0.6839$$

$\vdots$

$x_3 \approx 0.5775$

$x_4 \simeq 0.5672$

$x_5 \simeq 0.56714$

$x_6 \simeq 0.56714$

$x_7 = 0.56714$

Soln for $xe^x = 1$ is $x \simeq 0.56714$

Ω - Constant

Ω is a solution to function (omega constant)

$xe^x = 1$

$\Omega = 0.56714$

$\Rightarrow \Omega e^{\Omega} = 1$

Lambert-W function

Let $f(x) = xe^x$

(Cannot easily find function

Then $W(x) = f^{-1}(x)$

Solve : $xe^x = 1$, $f(x) = 1$

$W(f(x)) = W(1)$

$\downarrow$

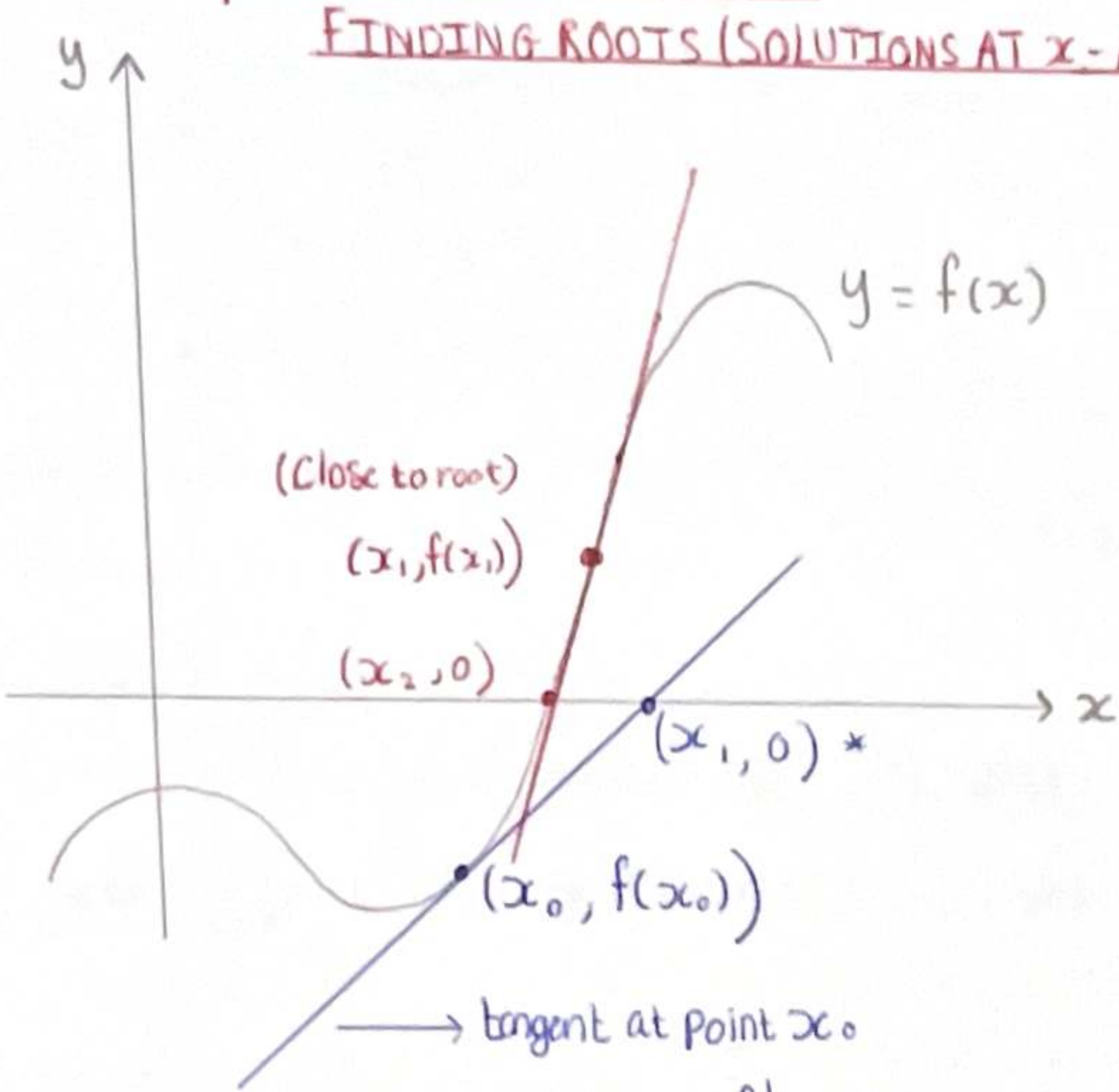
$f^{-1}(f(x)) = W(1)$

$\Rightarrow x = W(1)$

$$\boxed{W(1) = 0.56714 = \Omega}$$

Newton - Raphson Method

FINDING ROOTS (SOLUTIONS AT x - AXIS)



Slope of line $= f'(x_0)$

X Eqn of tangent : $y = mx + c$

from gradient $= \frac{y_2 - y_1}{x_2 - x_1}$

$$\Rightarrow \frac{y - f(x_0)}{x - x_0} = f'(x_0)$$

Solve for $x_0 = 0 \equiv x_1$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

Plug in $(x_1, 0)$

$$\Rightarrow 0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

$$x_1 - x_0 = \frac{-f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\boxed{x_{n-1} = x_n - \frac{f(x_{n-1})}{f'(x_{n-1})}}$$

or

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

① Choose a random point x_0

Consider another point x_1

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

More tangent lines drawn imply better root Approximation.

or

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\int_0^\infty \operatorname{sinc}(x)\,dx = \int_0^\infty \frac{\sin(x)}{x}\,dx \longrightarrow \int_0^\infty \frac{\sin x}{x}\underbrace{e^{-0x}}_{1}\,dx \quad \text{for } I(0) \text{ ie } b=0$$

Introduce Parameter b

Consider

$$\frac{\sin x}{x} e^{bx} = I(b)$$

$$\Rightarrow I(b) = \int_0^\infty \frac{\sin x}{x} \times e^{-bx}\,dx \quad — (\beta)$$

$$I'(b) = \frac{d}{db}\left(\int_0^\infty \frac{\sin x}{x} \times e^{-bx}\,dx\right)$$

differentiate w.r.t b

$$= \frac{d}{db} I(b)$$

differentiation under Integration.

$$\Rightarrow I'(b) = \int_0^\infty \frac{d}{db} \frac{\sin x}{x} e^{-bx}\,dx$$

Differentiate w.r.t b only $\Rightarrow$ Partial derivative

$$= \int_0^\infty \left(\frac{\partial}{\partial b} \frac{\sin x}{x} e^{-bx}\right) dx$$

x is a constant when differentiating w.r.t b

$$I'(b) = \int_0^\infty \frac{\sin x \cdot e^{-bx} x - x}{x}\,dx$$

$$= -\int_0^\infty \sin x \cdot e^{-bx}\,dx \quad (2)$$

b is a constant as we are integrating w.r.t x

LATE $\quad \int uv' = uv - \int u'v$

Consider $\int \sin(x) \cdot e^{-bx}\,dx$, Apply integration by Parts.

$u = \sin x \qquad v = -\frac{e^{-bx}}{b}$

$u' = \cos x \qquad v' = e^{-bx}$

$$\Rightarrow -\frac{\sin x\, e^{-bx}}{b} + \int \cos x \frac{e^{-bx}}{b} = -\frac{1}{b}\sin x\, e^{-bx} + \frac{1}{b}\int \cos x\, e^{-bx}\,dx \quad -(1)$$

Consider $\int \cos x \, e^{-bx} \, dx$

$U = \cos x \quad V = \frac{-e^{-bx}}{b}$

$U' = -\sin x \quad V' = e^{-bx}$

$\Rightarrow \frac{-\cos x \, e^{-bx}}{b} + -\int -\sin x \times \frac{-e^{-bx}}{b} \, dx$

$= -\frac{1}{b} \cos x \, e^{-bx} - \frac{1}{b}\int \sin x \, e^{-bx} \, dx$

$= -\frac{1}{b^2} \cos x \, e^{-bx} - \frac{1}{b^2}\int \sin x \, e^{-bx} \, dx$

$= -\frac{1}{b} \sin x \, e^{-bx} + -\frac{1}{b^2} \cos x \, e^{-bx}$

$= \cancel{\frac{-e^{-bx}(\cos x + \sin x)}{b^2+1}}$

$= \frac{-e^{bx}(\cos x + b\sin x)}{b^2+1}$

$\Rightarrow I'(b) = + \frac{+e^{-bx}(\cos x + b\sin x)}{b^2+1}\Bigg|_{x=0}^{x=\infty}$ – from (α)

Note b must be +ve for $e^{-\infty} = 0$ $\boxed{b \geq 0}$

else $= \frac{1}{e^{-\infty}} = e^{\infty} = \infty$ (No Convergence)

$a = b + c'$

$c' = d + a$

$a - c' = b$

$c' - a = d$

$a + c' - c' - a = b + d$

$I'(b)$ when $x \to \infty$

$\Rightarrow \frac{\underbrace{e^{-\infty}}_{0}(\cos(\infty) + b\sin(\infty))}{b^2+1}$

$\Rightarrow 0$

When $x = 0$

$= \frac{\underbrace{e^{0}}_{1}(\overset{\to 1}{\cos(0)} + b\overset{0}{\sin(0)})}{b^2+1}$

$= \frac{1}{b^2+1}$

$\Rightarrow I'(b)\Big|_{x=0}^{x=\infty} = I'(b)$ when $x=\infty$ $- I'(b)$ when $x = 0$

$= 0 - \frac{1}{b^2+1}$

$I'(b) = -\frac{1}{b^2+1}$

$$\int I'(b)\, db = \int \frac{-1}{b^2+1}\, db$$

$$\Rightarrow I(b) = -\tan^{-1}(b) + C$$

from $\textcircled{\beta}$ $I(b) = \int_0^\infty \frac{\sin x}{x} e^{-bx}\, dx$

$$\Rightarrow -\tan^{-1}(b) + C = \int_0^\infty \frac{\sin x}{x} e^{-bx}\, dx$$

~~Put b =~~

let $b \to \infty$

$$\underbrace{-\tan^{-1}(\infty)}_{} + C = \cancel{\int_0^\infty \frac{\sin(x)}{x}} \int_0^\infty \frac{\sin x}{x} \cdot \overbrace{e^{-\infty x}}^{0}\, dx$$

$$-\frac{\pi}{2} + C = \underbrace{\int_0^\infty 0\, dx}_{0} = 0 \quad \Rightarrow -\frac{\pi}{2} + C = 0$$

$$\Rightarrow C = \frac{\pi}{2}$$

$$\Rightarrow I(b) = -\tan^{-1}(b) + \frac{\pi}{2}$$

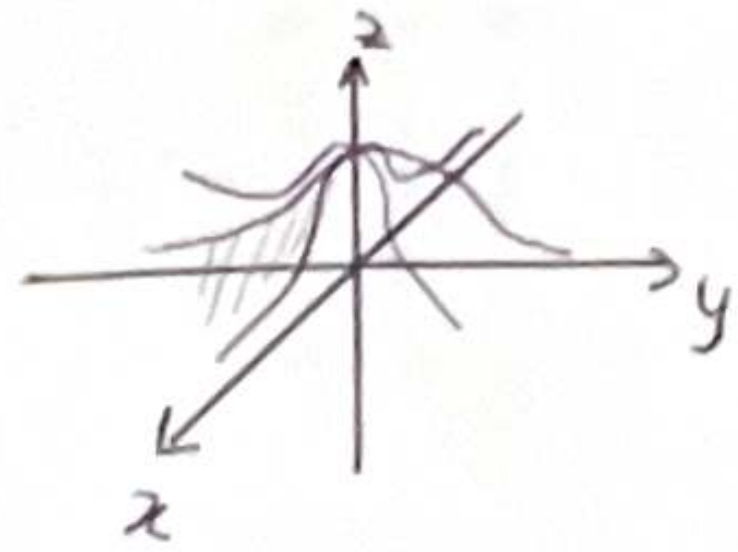
Find $I(0)$ which $= \int_0^\infty \frac{\sin x}{x} \underbrace{e^{-0x}}_{1}\, dx = \int_0^\infty \frac{\sin x}{x}\, dx$

$$I(0) = \underbrace{-\tan^{-1}(0)}_{0} + \frac{\pi}{2}$$

$$\Rightarrow I(0) = \frac{\pi}{2}$$

$$\Rightarrow \int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2} \quad \Rightarrow \int_0^\infty \operatorname{Sinc}(x) = +\frac{\pi}{2} \leftarrow$$

$$\iiint_{\mathbb{R}^3} e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz = I \longrightarrow$$

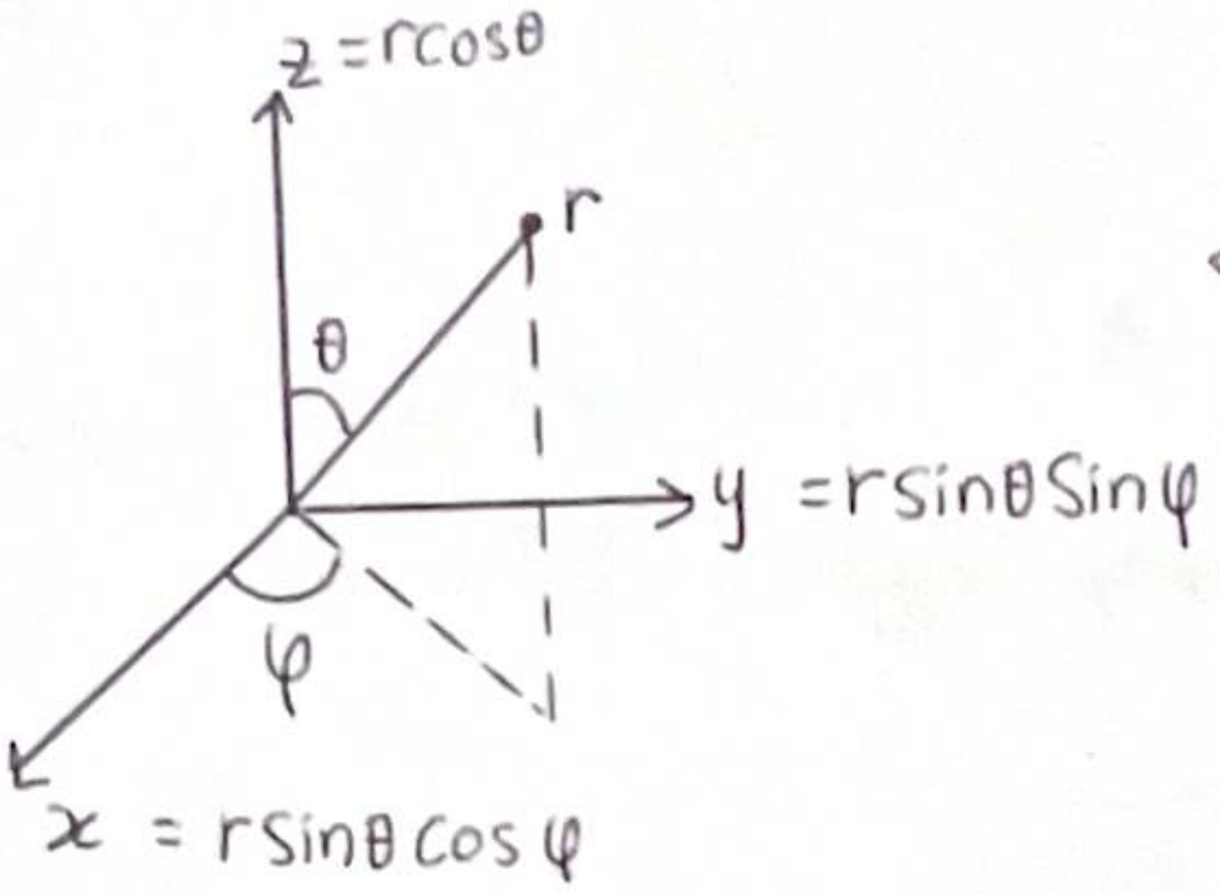


True ?? r is always +ve

$r \in [0, \infty)$

$\begin{cases} \theta \in (0, \pi) \\ \varphi \in (0, 2\pi) \end{cases}$

Condition for $r \geq 0$



$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

Sign before constant of each determinant

Note: $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{-r^2}\,dr\,d\theta\,d\varphi\,|J|$$

where $|J|$ is the Jacobian determinant

$$|J| = \det.\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} \overset{\det.}{=} \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta \begin{vmatrix} r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \end{vmatrix} \overset{+}{-} (-r\sin\theta) \begin{vmatrix} \sin\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\sin\theta\cos\varphi \end{vmatrix}$$

$$= \cos\theta\left(r^2\sin\theta\cos\theta\cos^2\varphi + r^2\sin\theta\cos\theta\sin^2\varphi\right) + r\sin\theta\left(r\sin^2\theta\cos^2\varphi + r\sin^2\theta\sin^2\varphi\right)$$

$$= r^2\sin\theta\cos^2\theta\cos^2\varphi + r^2\sin\theta\cos^2\theta\sin^2\varphi + r^2\sin^3\theta\cos^2\varphi + r^2\sin^3\theta\sin^2\varphi$$

$$= r^2\sin\theta\left[\cos^2\theta\cos^2\varphi + \cos^2\theta\sin^2\varphi + \sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi\right]$$

$$= r^2\sin\theta\left[\cos^2\theta(\underbrace{\cos^2\varphi + \sin^2\varphi}_{1}) + \sin^2\theta(\underbrace{\cos^2\varphi + \sin^2\varphi}_{1})\right]$$

$$= r^2\sin\theta\left[\underbrace{\cos^2\theta + \sin^2\theta}_{1}\right] = r^2\sin\theta \quad (= |J|)$$

$$\therefore I = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^2 e^{-r^2} \sin\theta \, dr \, d\theta \, d\varphi$$

Using Fubini's Theorem

$$I = \int_0^{\infty} r^2 e^{-r^2} dr \times \underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{\downarrow} \times \underbrace{\int_0^{2\pi} d\varphi}_{\downarrow\ 2\pi} \qquad \therefore I = 4\pi \underbrace{\int_0^{\infty} r^2 e^{-r^2} dr}_{\text{Let this} = J} \quad *$$

$$\left[-\cos\theta\right|_0^{\pi}$$

$$= \left[-(-1) - (-1)\right]$$

$$= (1+1)$$

$$= 2$$

Differentiate under Integral Sign.

$$\frac{d^2}{dr^2} e^{-r^2} = \frac{d}{dr} -2re^{-r^2}$$

Let $U = -2r$ $\quad V = e^{-r^2}$

$U' = -2$ $\quad V' = -2re^{-r^2}$

$$\therefore \frac{d}{dr} -2re^{-r^2} = 4r^2e^{-r^2} - 2e^{-r^2}$$

Improper integral.

$$\therefore \int_0^{\infty} \frac{d^2}{dr^2} e^{-r^2} dr = \frac{d^2}{dr^2} \int_0^{\infty} e^{-r^2} dr = 0 \text{ (Derivative of a Constant = 0)}$$

Constant.

$$\therefore 0 = \int_0^{\infty} 4r^2e^{-r^2} - 2e^{-r^2} dr$$

$$\therefore \int_0^{\infty} 2e^{-r^2} dr = \int_0^{\infty} 4r^2e^{-r^2} dr$$

$$\therefore \int_0^{\infty} \frac{2}{4} e^{-r^2}\, dr = \underbrace{\int_0^{\infty} r^2 e^{-r^2}\, dr}_{J}$$

$$\cancel{\therefore J = \frac{1}{2}\int_0^{\infty} e^{-r^2}\, dr \rightarrow \sqrt{\pi}\ \text{(Gaussian integral)}}$$

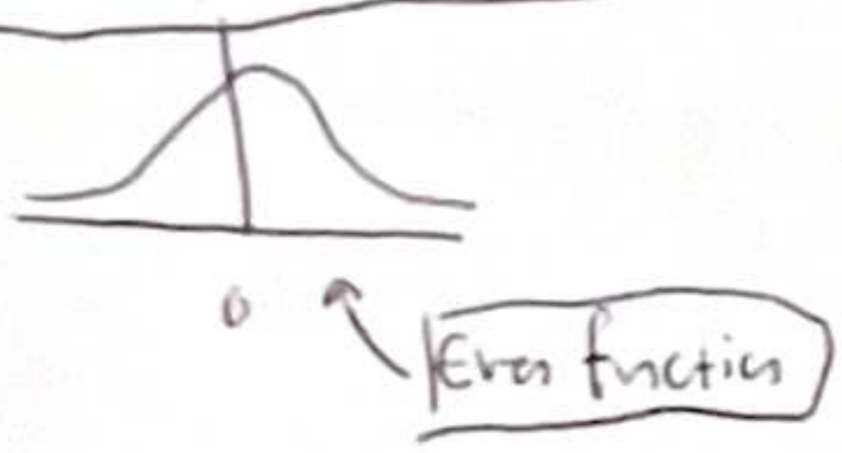
$$\cancel{\therefore J = \frac{\sqrt{\pi}}{2}}$$

$$\ast\ \therefore \boxed{I = \iiint_{\mathbb{R}} e^{-(x^2+y^2+z^2)}\, dx\,dy\,dz = \pi\sqrt{\pi}}$$

$$J = \frac{1}{2}\int_0^{\infty} e^{-r^2}\, dr$$

[Gaussian Identity]

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}}$$



$$\left.\begin{aligned} \therefore \int_{-\infty}^{\infty} f(x)\, dx &= 2\int_0^{\infty} f(x)\, dx \\ \int_0^{\infty} f(x)\, dx &= \frac{1}{2}\int_{-\infty}^{\infty} f(x)\, dx \end{aligned}\right\}$$ where $f(x)$ is an even function.

$$\therefore J = \frac{1}{2} \times \frac{1}{2}\int_0^{\infty} e^{-r^2}\, dr \rightarrow \sqrt{\pi}\ \text{(Gaussian Integral)}$$

$$J = \frac{\sqrt{\pi}}{4}$$

$$I = 4\pi \underbrace{\int_0^{\infty} r^2 e^{-r^2}\, dr}_{= J = \frac{\sqrt{\pi}}{4}}$$

$\ast$

$$\therefore I = 4\pi \times \frac{\sqrt{\pi}}{4} = \pi^{3/2} = \boxed{\pi\sqrt{\pi}} \leftarrow$$

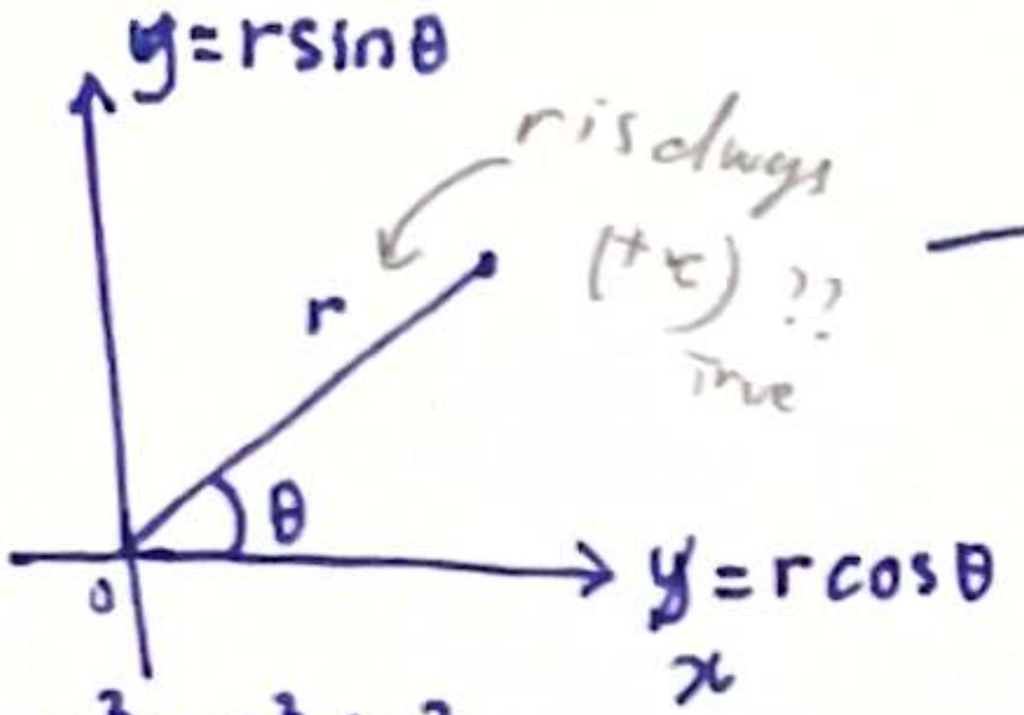
Show that: $\boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$ ← Gaussin Integral.

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$I = \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$\mathbb{R}^2 \rightarrow$ Pol



$r^2 = x^2+y^2$

range / freedom of variables

$\rightarrow$ $r \in [0,\infty)$ $\quad \theta \in [0, 2\pi]$

ie: $0 \leq r < \infty$ $\quad 0 \leq \theta \leq 2\pi$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy \longrightarrow I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,dr\,d\theta\,|J|$$

Where $|J|$ is the Jacobian determinant for (I^2 involving with) $y = r\sin\theta$, $x = r\cos\theta$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = (r\cos^2\theta) - (-r\sin^2\theta)$$

$$= r\,(\underbrace{\cos^2\theta + \sin^2\theta}_{1})$$

$$= r$$

$$\therefore I^2 = \int_0^{2\pi}\int_0^{\infty} r e^{-r^2}\,dr\,d\theta \longrightarrow \theta \text{ is not present in integrand}$$

$$\therefore I^2 = 2\pi\int_0^{\infty} r e^{-r^2}\,dr$$

let $u = r^2$

$$\frac{du}{dr} = 2r$$

$$dr = \frac{1}{2r}\, du$$

$$I^2 = 2\pi \int_0^\infty r e^{-r^2} dr$$

$$\downarrow u = r^2$$

$$= 2\pi \int_0^\infty r e^{-u} du \, |J|$$

where $|J| = \left|\frac{\partial u}{\partial r}\right| = 2r$??

u-substitution ≡ Jacobian for 1 eqn ?

* $\frac{k}{0} = \infty$

$\frac{k}{\infty} = 0$

$$\cancel{\therefore I^2 = 2\pi \int_0^\infty r e^{-u^2} \cdot \frac{1}{2r} du}$$

$$\therefore I^2 = \cancel{2}\pi \int_0^\infty \cancel{r} e^{-u} \cdot \frac{1}{\cancel{2}\cancel{r}} du$$

$\longrightarrow$ $u = r^2$, $r = \pm\sqrt{u}$ $\rightarrow$ $I^2 = \cancel{2}\pi \int_0^\infty \cancel{\pm\sqrt{u}}\, e^{-u} \frac{1}{\cancel{2}\cancel{\pm\sqrt{u}}} du$

$$= \pi \int_0^\infty e^{-u} du$$

$$= \pi \int_0^\infty e^{-u} du$$

$$= \pi \left[\frac{e^{-u}}{-1} \right|_0^\infty = \pi \left[-e^{-u} \right|_0^\infty$$

$$= \pi \left[(-\underbrace{e^{-\infty}}_{0}) - \underbrace{(-e^{0})}_{+\ -1} \right]$$

$\therefore = +1$

$$= \pi$$

$$I^2 = \pi$$

$$I = \pm\sqrt{\pi}$$

A·U·G is always a +ve value

$\therefore -\sqrt{\pi}$ is rejected

$$\therefore \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \longleftarrow \text{(shown)}$$

$\int x^i \, dx =$

$$\boxed{\begin{aligned} x^i &= \left(e^{\ln x}\right)^i = e^{i\ln x} \\ \text{or } x^i &= e^{\ln(x^i)} = e^{i\ln(x)} \end{aligned}} = \boxed{\cos(\ln x) + i\sin(\ln x)} \rightarrow x = e^{\ln x}$$

vv

$\approx \cos$

$$\int x^i \, dx = \frac{x^{i+1}}{i+1} + c \quad dx \leftarrow$$

vv

$$\int \cos(\ln x) + i\sin(\ln x) = \underbrace{\int \cos(\ln x)\,dx}_{Re} + i\underbrace{\int \sin(\ln x)\,dx}_{Im} \leftarrow (1)$$

Consider: $\dfrac{1(1-i)}{(1+i)(1-i)} \times x \times x^i = \dfrac{(1-i)}{2} \times x \times x^i = \dfrac{x}{2} \times (1-i) \times \underbrace{x^i}_{(1)}$

(also the soln for $\int x^i\,dx$)

$1^2 - (\sqrt{-1})^2 = 2$

$$= \frac{x}{2} \times (1-i) \times \left[\cos(\ln x) + i\sin(\ln x)\right]$$

Distribute

$$\frac{x}{2}\left[\cos(\ln x) + i\sin(\ln x) - i\cos(\ln x) + \sin(\ln x)\right]$$

$$= \frac{x}{2}\left[\underbrace{\cos(\ln x) + \sin(\ln x)}_{Re} + i\underbrace{\left(\sin(\ln x) - \cos(\ln x)\right)}_{Im}\right]$$

$$\Rightarrow \int \cos(\ln x)\,dx = \frac{x}{2}\left(\cos(\ln x) + \sin(\ln x)\right) + C_1$$

$$\int \sin(\ln x)\,dx = \frac{x}{2}\left(\sin(\ln x) - \cos(\ln x)\right) + C_2$$

$$\Rightarrow \int x^i\,dx = \frac{x}{2}\left(\cos(\ln x) + \sin(\ln x)\right) + C_1 + i\left[\frac{x}{2}\left(\sin(\ln x) - \cos(\ln x)\right) + C_2\right] \leftarrow$$

① $\cancel{\int_0^1 e}$ $\int_0^1 x^{-x}\,dx$ $\cancel{= \int_0^1 x^{\ln}}$

$$= \int_0^1 e^{\ln(x^{-x})}\,dx$$

$$= \int_0^1 e^{-x\ln(x)}\,dx$$

FROM $\sum_{k=0}^{\infty} \frac{y^k}{k!} = e^y$

$e^{-x\ln(x)} = e^y \Rightarrow y = -x\ln(x)$

$$= \int_0^1 \sum_{k=0}^{\infty} \frac{(-x\ln(x))^k}{k!}\,dx$$

$$= \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^k \cdot \ln^k(x)}{k!}\,dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 x^k \cdot \ln^k(x)\,dx$$

Put $u = \ln(x)$

$\cancel{du = \frac{1}{x}\,dx}$

$\cancel{\Rightarrow x = e^u}$

$\cancel{x = e^u - ①}$

find limits

$x = 0$

$u = \ln(0) = -\infty$

$x = 1$

$u = \ln(1) = 0$

$x = e^{-\infty} = 0$

X

$x = -e^u - ①$

~~Put~~ $-u = \ln x \Rightarrow$

$\cancel{-du = \frac{1}{x}\,dx}$ ~~New limits~~ $\cancel{x = 0}$

$\cancel{-x\,du = dx}$

~~From ①~~

$\cancel{-e^u\,du} = dx$

Put $-u = \ln x$

$x = e^{-u}$ — ① $\rightarrow$ New limits

$x = 0$

$-u = -\infty$

$u = \infty$

$x = 1$

$-u = 0$

$u = 0$

$-du = \frac{1}{x}\,dx$

$-x\,du = dx$

$-e^{-u}\,du = dx$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{\infty}^{0} e^{-uk} \cdot (-u)^k \cdot -e^{-u}\,du$$

$\cancel{= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int}$ Use $\int_a^b = -\int_b^a$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^{\infty} e^{-uk} \cdot (-u)^k \cdot e^{-u}\,du$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^{\infty} e^{-uk} \cdot (-1)^k \cdot u^k \cdot e^{-u}\,du$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k(-1)^k}{k!} \int_0^{\infty} e^{-uk} \cdot u^k \cdot e^{-u}\,du$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{2k}}{k!} \int_0^{\infty} e^{-uk} \cdot u^k \cdot e^{-u}\,du$$

even power of -1 always $= 1$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^{\infty} e^{-uk} \cdot u^k \cdot e^{-u}\,du$$

$$e^{-uk} \times e^{-u} = e^{-u(k+1)}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^{\infty} u^k \cdot e^{-u(k+1)}\,du$$

Consider $\Gamma(z) = (z-1)! = \int_0^\infty e^{-t} . t^{z-1} \, dt$

Put $K = (z-1)$

$\Rightarrow K! = \int_0^\infty e^{-t} . t^{K} \, dt$

$$\rightarrow \sum_{K=0}^{\infty} \frac{1}{K!} \int_0^\infty e^{-U(K+1)} . U^K \, dU$$

Put $V = \underbrace{U(K+1)}_{\text{constand}}$ $\longrightarrow$ New limits
$U = 0, V = 0$
$U = \infty, V = \infty$

$dV = (K+1)\, dU$

$U = \frac{V}{K+1}$

$dU = \frac{dV}{K+1}$

$$= \sum_{K=0}^{\infty} \frac{1}{K!} \int_0^\infty e^{-V} . \left(\frac{V}{(K+1)}\right)^K . \frac{dV}{K+1}$$

$$= \sum_{K=0}^{\infty} \frac{1}{K!} \int_0^\infty e^{-V} . \frac{V^K}{(K+1)^K} . \frac{dV}{(K+1)}$$

$\gamma = (K+1) \Rightarrow \gamma^K \times \gamma = \gamma^{K+1}$

$$\Rightarrow \sum_{K=0}^{\infty} \frac{1}{K!} \int_0^\infty e^{-V} . V^K . \underbrace{\frac{1}{(K+1)^{K+1}}}_{\text{constant}} \, dV$$

$$= \sum_{K=0}^{\infty} \frac{1}{K!} \frac{1}{(K+1)^{K+1}} \underbrace{\int_0^\infty e^{-V} . V^K \, dV}_{K!}$$

$K! = \int_0^\infty e^{-V} . V^K \, dV$

from gamma function

$$= \sum_{K=0}^{\infty} \frac{1}{\cancel{K!}} \times \cancel{K!} \times \frac{1}{(K+1)^{K+1}}$$

$$= \sum_{K=0}^{\infty} \frac{1}{(K+1)^{K+1}} \longleftarrow$$

$\alpha = K+1$
when $K = 0, \alpha = 1$
when $K = \infty, \alpha = \infty$

$$\Rightarrow \sum_{\alpha=1}^{\infty} \frac{1}{\alpha^\alpha} \longleftarrow$$

$$\Rightarrow \sum_{K=0}^{\infty} \frac{1}{(K+1)^{K+1}}$$

$$= \sum_{K=0}^{\infty} (K+1)^{-(K+1)}$$

$\cancel{y = 5x+3}$

$\cancel{\sum_{k=0}^{\infty} \frac{x}{k!} + \frac{3}{k!}}$

$$\boxed{\int_0^1 x^{-x}\,dx}$$

$$\int_0^1 x^{-x}\,dx = \int_0^1 e^{\ln(x^{-x})}\,dx = \int_0^1 e^{-x\ln x}\,dx$$

$$= \int_0^1 \sum_{k=0}^{\infty} \frac{(-x(\ln x))^k}{k!}\,dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^k \ln^k(x)}{k!}\,dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 x^k \ln^k(x)\,dx \leftarrow$$

let $u = \ln(x)$

$\cancel{\frac{du}{dx} = \frac{1}{x}}$ $\cancel{\text{If } u = 0}$, If $x = 0$, then $u = -\infty$

NO convergence

$\int_{-\infty}^{\infty}$ ✗

Let $-u = \ln(x)$ - ①

$\therefore -du = \frac{1}{x}\,dx$

$-x\,du = dx$

From ① $x = e^{-u}$

$\Rightarrow -e^{-u}\,du = dx$

New limits

$-u = \ln(0) = -(-\infty) = \infty$

$-u = \ln(1) = -0 = 0$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{\infty}^{0} e^{-uk} \cdot (-u)^k \cdot -e^{-u}\,du$$

Use property $\int_a^b = -\int_b^a$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^{\infty} e^{-uk} \cdot (-1)^k \cdot u^k \cdot e^{-u}\,du$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k(-1)^k}{k!} \int_0^{\infty} e^{-uk} \cdot u^k \cdot e^{-u}\,du$$

-1 to even power = 1

$$= \sum_{k=0}^{\infty} \frac{(-1)^{2k}}{k!} \int_0^{\infty} e^{-uk} \cdot u^k \cdot e^{-u}\,du$$

$$= \sum_{K=0}^{\infty} \frac{1}{K!} \int_0^{\infty} e^{-UK} \cdot U^K \cdot e^{-U} \, dU = \sum_{K=0}^{\infty} \frac{1}{K!} \int_0^{\infty} e^{-U(K+1)} \cdot U^K \, dU$$

$$e^{-UK} \times e^{-U} = e^{-U(K+1)}$$

Consider Gamma function.

$$\Gamma(z) = (z-1)! = \int_0^{\infty} e^{-t} t^{z-1} \, dt$$

Let $V = U(\overset{\text{constant}}{K+1})$

$dV = (K+1)\, dU$

$dU = \dfrac{dV}{K+1}$

new limits
$U = 0, V = 0$
$U = \infty, V = \infty$

$$= \sum_{K=0}^{\infty} \frac{1}{K!} \int_0^{\infty} e^{-V} \cdot \left(\frac{V}{(K+1)}\right)^K \frac{dV}{(K+1)} = \sum_{K=0}^{\infty} \frac{1}{K!} \int_0^{\infty} e^{-V} \cdot \frac{V^K}{(K+1)^K} \cdot \frac{dV}{(K+1)}$$

$\alpha = K+1$

$\alpha^K \times \alpha = \alpha^{K+1}$
$= (K+1)^{K+1}$

$$= \sum_{K=0}^{\infty} \frac{1}{K!} \frac{1}{(K+1)^{K+1}} \underbrace{\int_0^{\infty} e^{-V} V^K \, dV}_{K!}$$

$\longrightarrow$ from Γ function

$K! = \int_0^{\infty} e^{-t} t^K$ $\quad \frac{1}{K!} \times K! = 1$

$$= \sum_{K=0}^{\infty} \frac{1}{(K+1)^{K+1}}$$

let $\alpha = K+1$

$$\Rightarrow \sum_{\alpha=1}^{\infty} \frac{1}{\alpha^{\alpha}}$$

$$e^{\ln(2)} = 2 \qquad e^{\ln(x)} = x$$

$$y = e^{\ln(x)}$$

$$\ln y = \ln e^{\ln(x)}$$

$$\ln y = \ln(x) \cdot \underbrace{\ln(e)}_{1}$$

$$\Rightarrow y = x$$

$$e^{\ln(x)} = x \leftarrow$$

$$\sum_{n=0}^{\infty} \frac{y^n}{n!} = e^y \quad y \in \mathbb{R} \ (\text{or } y \in \mathbb{C})$$

$$\rightarrow \frac{y^0}{0!} + \frac{y^1}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \frac{y^5}{5!} + \dots$$

$$y = 1$$

$$\rightarrow \frac{1^0}{0!} + \frac{1^1}{1!} + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!}$$

$$= 2.758 \leftarrow \approx \neq e^1 \simeq e$$

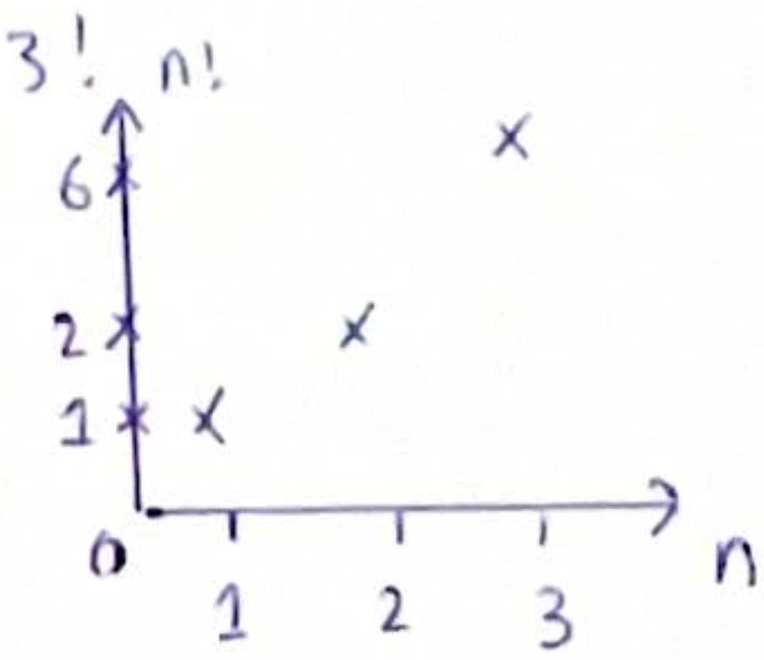
Gamma function Γ

It is an extension of the factorial function with its argument shifted down by 1, to real & complex nºs.

$\Gamma(n) = (n-1)!$ for $\cancel{\{n \mid n > 0\}}$

factorial function ($n!$) $\rightarrow \{n \mid n \in \mathbb{N}\}$

eg : $3!$



Cannot join line

$$F \cdot \ \Gamma(n) = \int_0^{\infty} x^{(n-1)} e^{-x} \, dx$$

~~for {n | n ∈ ℝ~~

for $\{n \mid n \in \mathbb{C} \wedge \mathrm{Re}(n) > 0\}$

Gamma function interpolates the factorial function to non integer values.

Formal Mathematics

• Mathematical Proofs

- A proposition is a sentence which is either true or false but not both.

- A statement and its contrapositive are either both true or both false so that $p \rightarrow q$ and $\sim p \rightarrow \sim q$ are equivalent statements.

① Direct proof by deduction.

Axiomatic method

This type of proof is accepted because ~~be~~ it is derived by correct implications and sound arguments from axioms.

- To prove by deduction that $p \rightarrow q$ is true, start with p then deduce $p \rightarrow r \rightarrow s \rightarrow q$ so $p \rightarrow q$ is true.

This is a proof of the implication $p \rightarrow q$ is true.

To prove: if $x^2+2x-3=0$ then $x=-3$ or 1 is true, we proceed as follows:

$$x^2+2x-3=0$$
$$\Rightarrow (x+3)(x-1)=0$$
$$\Rightarrow x+3=0 \quad \text{or} \quad x-1=0$$
$$x=-3 \quad \text{or} \quad x=1$$

Hence $x^2+2x-3=0$ does imply $x=-3$ or 1

Q.E.D

- Proof by induction

Steps: ① Let P_n be a statement involving n where n is any positive integer
② Proof directly the implication $P_K \Rightarrow P_{K+1}$ is correct
③ Combine Step ① & ② to show that $P_2, P_3, \ldots P_n$ are correct.
↖ usually expressed as 'therefore, by induction'

Prove by induction that $9^n - 1$ is a multiple of 8 for all positive integral values of n.

Let P_n be the statement "for all positive integral values of n, $9^n - 1$ is a multiple of 8."

If the formula is valid when $n = K$, ie $9^K - 1 = 8a$

then,
$$\begin{aligned} 9^{K+1} - 1 &= (9^K)(9) - 1 \\ &= (9^K)(9) - 9 + 8 \\ &= 9(9^K - 1) + 8 \\ &= 9(8a) + 8 \\ &= 8(9a + 1) \text{ which is a multiple of 8} \end{aligned}$$

ie. if $9^K - 1$ is a multiple of 8, So is $9^{K+1} - 1$

but if $n = 1$, $9^1 - 1 = 8 \Rightarrow P_1$ is true

$\Rightarrow P_2$ is true and So is P_3

So, $(\forall_n \in \mathbb{N})$ P_n is correct.

Q.E.D

Prove by induction that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$ for all positive integral values of n.

Let P_n be the statement $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$

If the formula is valid when $n=k$, ie. $\sum_{r=1}^{k} r^2 = \frac{k}{6}(k+1)(2k+1)$

then adding $(k+1)^2$ to both sides
(as $(k+1)$ is an additional term which can be added in the summation)

from $r^2 = (k+1) \Rightarrow r^2 = (k+1)^2$

$$\sum_{r=1}^{k+1} r^2 = \frac{k}{6}(k+1)(2k+1) + (k+1)^2$$

$$= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)^2]$$

$$= \frac{1}{6}(k+1)(2k^2+7k+6)$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

which is the formula when $n = k+1$

$\Rightarrow \therefore P_n$ is valid when $n=k$ which is also valid when $n=k+1$

When $n=1$

L.H.S of $P_n = 1$

R.H.S of $P_n = \frac{1}{6}(1+1)(2+1) = 1$

So when $n=1$, P_n is valid

$\therefore P_2, P_3, \ldots$ are also valid.

Use proof by induction to show that $\frac{d}{dx} x^n = n x^{n-1}$

Let P_n be the statement " If n is any positive integer, then $\frac{d}{dx}(x^n)\ n x^{n-1}$

If the formula is valid when $n = k$, ie $\frac{d}{dx} x^k = k x^{k-1}$

~~then~~ $\cancel{x^{k+1} = \frac{d}{dx} (k+1)}$ (

then when $x = k+1$, $\frac{d}{dx}(x^{k+1}) = \underbrace{\frac{d}{dx}(x^k)(x)}$

Using product rule,

$\frac{d}{dx}(x^k)(x) = (x^k)(1) + (kx^{k-1})(x)$

$= (k+1)(x^k)$ ← which is the formula when $n = k+1$

Since formula is valid when $n = k$, it is also valid when $n = k+1$

Now, if $n = 1$, from first principles,

$\frac{d}{dx}(x^1) = \lim_{\delta x \to 0}\left[\frac{(x+\delta x)-x}{\delta x}\right] = 1$

$= 1(x^0)$

Hence P_1 is true

by induction, so is $P_2, P_3, \ldots$

Prove by induction that for $n \in \mathbb{N}$ $n > 1$

From triangle inequality theorem,

$$|z_1 + z_2 + \dots + z_n| \leqslant |z_1| + |z_2| + \dots + |z_n|$$

Let P_n be the statement $|z_1 + z_2 + \dots z_n| \leqslant |z_1| + |z_2| + \dots + |z_n|$

~~$\forall n \in$~~ for $\{ \forall n \mid n > 1 \}$

If the inequality is valid when $n = k$

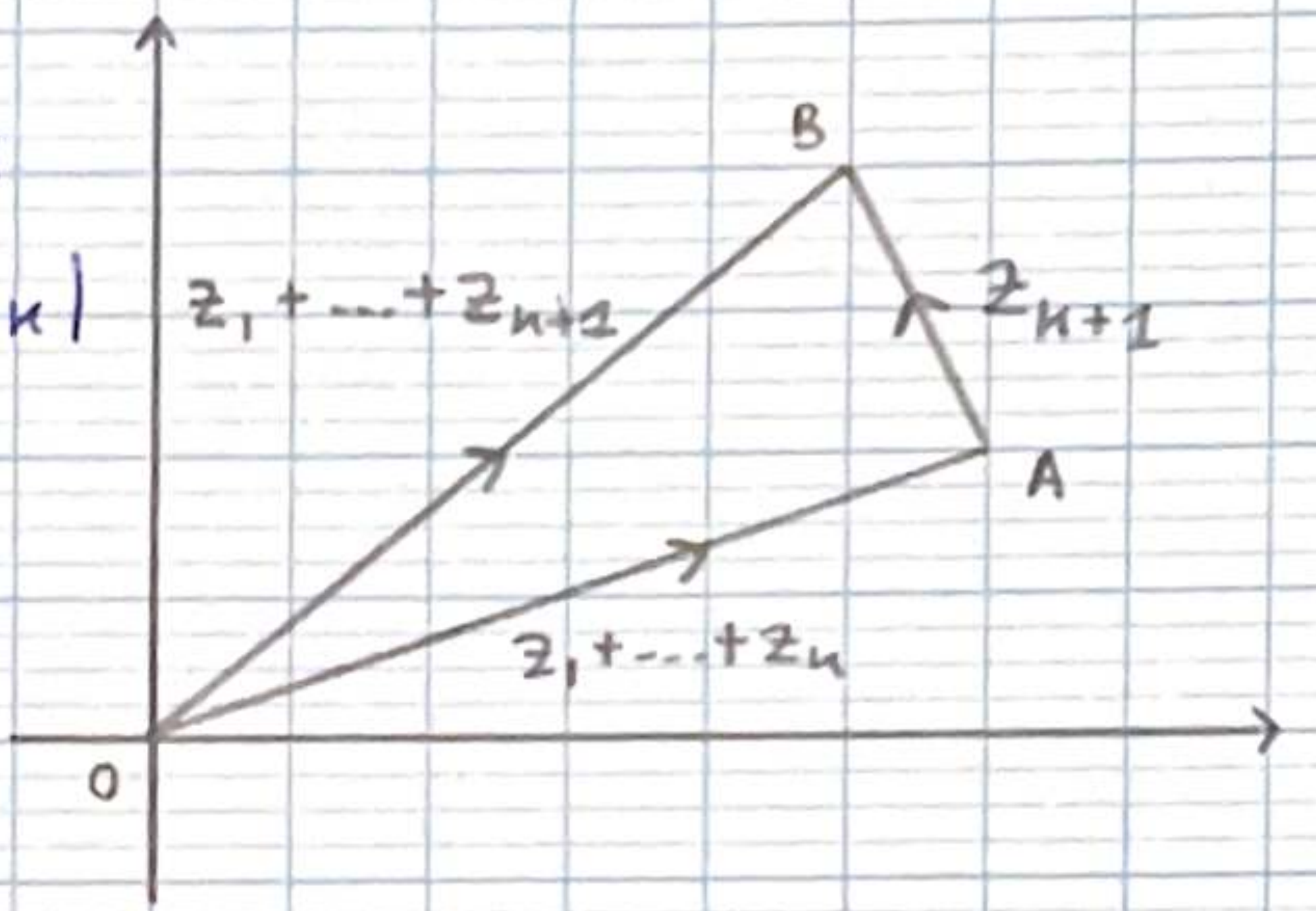
ie. $|z_1 + \dots + z_k| \leqslant |z_1| + \dots + |z_k|$ ①

then from diagram,

Let $OA = z_1 + \dots + z_k$

$OB = |z_1| + \dots + |z_k| + |z_{k+1}|$

$AB = OB - OA = |z_{k+1}|$



The points OAB form either a triangle or a line.

So, $|\vec{OB}| \leqslant |\vec{OA}| + |\vec{AB}|$

ie. $|z_1| + \dots + |z_k| + |z_{k+1}| \leqslant |z_1 + \dots + z_k| + |z_{k+1}|$

But if ① is true, then,

$|z_1 + \dots + z_{k+1}| \leqslant |z_1| + |z_2| + \dots + |z_k| + |z_{k+1}|$

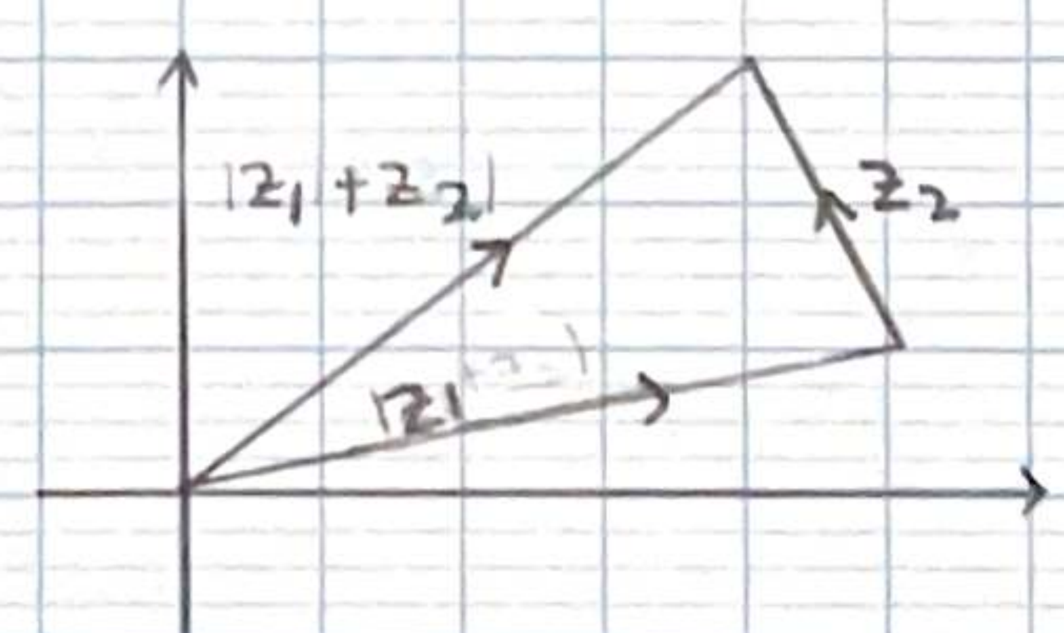
So if the inequality when $n = k$ is valid, then it is valid when $n = k+1$

when $n = 2$

Using triangle inequality ~~theorem~~ property

LHS of $P_n = |z_1 + z_2|$

RHS of $P_n = |z_1| + |z_2|$



From diagram

$\Rightarrow |z_1 + z_2| \leqslant |z_1| + |z_2|$

So P_n is valid when $n=2$

Therefore, by induction P_n is true for $n \in \mathbb{N}$ $n > 1$