Fluid dynamics & the Navier-Stokes equation

These are equations which can be used ~~the~~ to determine the velocity vector field that applies to a fluid given some initial conditions.
They arise from the application of Newton's Second law in combination with a fluid stress (due to viscosity) and a pressure term.
For almost all real situations, they result in a system of non-linear partial differential equations, however, with certain simplifications, (such as 1-D motion), they can sometimes be reduced to linear differential equations.
Usually, however, they remain non-linear which make them difficult or impossible to solve; this is what causes the turbulence and unpredictability in their results.

Continuity equation

eg. Intensive property: density
corresponding extensive property: mass.

The basic continuity equation is an equation which describes the change of an intensive property L. An intensive property is something which is independent on the total amount of material present. (eg. Temperature is an intensive property) Heat would be the corresponding extensive property. The volume Ω is assumed to be of any form; its bounding surface is referred as $\partial\Omega$. The continuity equation derived can later be applied to mass & momentum.

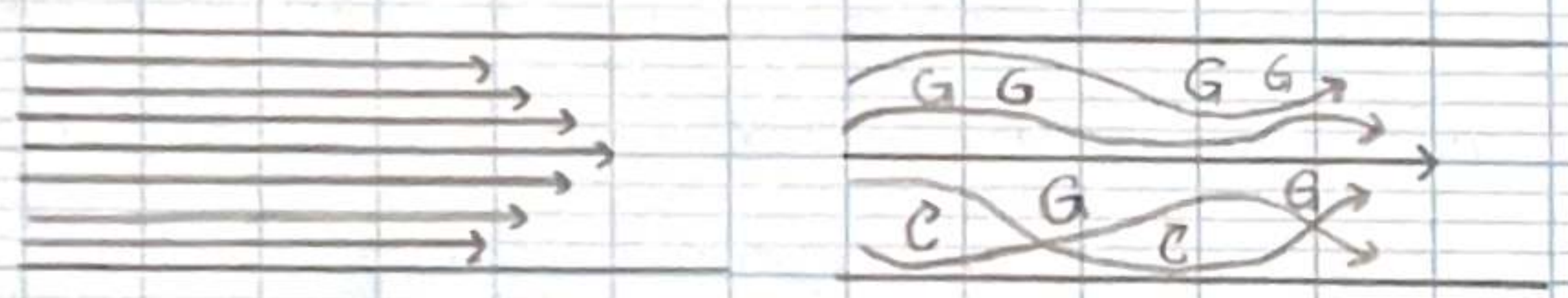
Boundary Condition Vs initial conditions. Chaos Theory.

Also given some initial conditions

The Navier-Stokes Equations are also of great interest in a purely mathematical sense. Unfortunately, the highly intermittent and irregular character of turbulent flow complicates all analyses. It has not yet been proven that in three dimensions, solutions always exist, or that if they do exist, then they are smooth. They yield Navier-Stokes equations with turbulence.
This is called the Navier-Stokes existence and Smoothness problem.

If smooth exists, they have banded energy.

Characteristics of turbulent flow.



Laminar flow Turbulent flow

- Turbulent flow tends to occur at higher velocities, low viscosity and at higher characteristic linear dimensions.
- If the Reynolds number is greater than $Re > 3500$, the flow is turbulent.

Why is 3500 critical?

$$Re = \frac{\text{Inertia forces}}{\text{Viscous forces}} = \frac{\rho . V . D}{\mu}$$

$\uparrow Re \rightarrow \uparrow$ Velocity
$\uparrow$ characteristic dimension
$\uparrow$ Density
$\downarrow$ Viscosity.

- Irregularity: The flow is characterised by irregular movement of particles of the fluid. The movement of fluid particles is chaotic. For this reason, turbulent flow is normally treated statistically rather than deterministically.
- Diffusity: In turbulent flow, a fairly flat ~~Section~~ velocity distribution exists across the section of a pipe, with the result that the entire fluid flows at a given single value and drops rapidly extremely close to the walls. The characteristic which is responsible for the enhanced mixing and increased rates of mass, momentum and energy transports in a flow is called "diffusivity".
- Rotationality: Turbulent flow is characterised by strong three-dimensional vortex generation mechanism. This mechanism is known as vortex stretching.
- Dissipation: A dissipative process is a process in which the kinetic energy of turbulent flow is transformed into internal energy by viscous shear stress.

Derivation of the Navier-Stokes equation.

The navier-stokes equation can be derived from the basic conservation and continuity equations applied to properties of fluids.

Reynold's transport theorem

$$\underbrace{\frac{d}{dt}\int_{\Omega} L \, dV}_{} = \underbrace{-\int_{\partial\Omega} L\vec{V}.\vec{n} \, dA}_{} \underbrace{-\int_{\Omega} Q \, dV}_{}$$

This part of the equation denotes the rate of change of the property L contained inside the volume Ω.

It is a flux term which indicates how much of the property L is leaving the volume by flowing over the bondary $\partial\Omega$.

A sink term which indicates how much of the property L is leaving the volume due to sinks or sources inside the boundary.

The above equation states that the change in the total amount of a property is due to how much flows out through the volume boundary as well as how much is lost or gained through sources or sinks inside the boundary.

For the intensive property density, the equation is a statement of the conservation of mass, total change in mass, is the sum of what leaves the boundary and what appears within it (mass source or sink in the volume); no mass is left unaccounted for.
??

Divergence Theorem.

The divergence theorem allows the flux term of the above equation to be rexpressed as a volume integral.

By the Divergence Theorem,

$$\int_{\partial\Omega} L\vec{V}.\vec{n} \, dA = \int_{\Omega} \nabla.(L\vec{V}) \, dV$$

The previous equation can therefore be rewritten as,

$$\frac{d}{dt}\int_{\Omega} L \, dV = -\int_{\Omega} \underbrace{\nabla\cdot(L\vec{V}) + Q}_{} \, dV$$

Q & $\nabla\cdot(L\vec{V})$ have common integral and can be grouped together.

Leibniz's Rule States that

$$\frac{d}{dx}\int_a^b f(x,y)\,dy = \int_a^b \frac{d}{dx} f(x,y)\,dy$$

Applying this rule to the previous equation yields

$$\int_{\Omega} \frac{d}{dt} L \, dV = -\int_{\Omega} \nabla \cdot (L\vec{V}) + Q \, dV$$

Equivalently,

$$\int_{\Omega} \frac{d}{dt} L \, dV + \nabla \cdot (L\vec{V}) + Q \, dV = 0 \leftarrow$$ Grouping integrals after sending to LHS.

The above equation applies to any control volume Ω, the only way the equation remains true for all control volumes is if the integrand itself is zero. If integral is zero, $\Rightarrow$ Integrand $= 0$. Thus we arrive to the general form for continuity equation.

$$\boxed{\frac{dL}{dt} + \nabla \cdot (L\vec{V}) + Q = 0}$$

Conservation of mass

Applying the continuity equation to density (the intensive property equivalent to mass), we obtain

$$\frac{d\rho}{dt} + \nabla \cdot (\rho\vec{V}) + Q = 0$$

This is the same conservation of mass because we are operating with a constant control volume Ω. With no source or sink of mass ($Q = 0$)

$$\frac{d\rho}{dt} + \nabla \cdot (\rho\vec{V}) = 0 \leftarrow$$ This is the equation of conservation of mass.

In some cases such as where we have a pipe pumping fluid in or out of the system inside the control volume, we might not want to assume $Q = 0$, however, for general cases we make the assumption no mass is added or removed from the system. However in certain cases it is useful to simplify futher. For an incompressible fluid, density is constant. Setting the derivative of density equal to zero and dividing through by a constant ρ we obtain the simplest form of the equation:

$$\boxed{\nabla \cdot \vec{V} = 0}$$

$\int y' \, dx$?!

grad = y/x

y'

$\int \frac{y}{x} dx$

$x \to 0 = dx$

$\int \frac{y}{dx} \times dx$

$= \int y =$ continuous summation of y $\equiv$ original property

Material Derivative

A normal derivative is the rate of change of an intensive property at a point. For instance, $\frac{dT}{dt}$ could be the rate of change of temperature at point (x, y). However, a material derivative is the rate of change of an intensive property on a particle in a velocity field. It must incorporate 2 things:

- Rate of change of the property, $\frac{dL}{dt}$
- change in position of the particle in the velocity field $\vec{V}$.

The material derivative can be defined as: $\frac{DU}{Dt} = \frac{dU}{dt} + (\vec{V} \cdot \nabla)U$

This expression represents the directional derivative of U in the direction of the velocity $\vec{V}$.

Conservation of momentum.

This derivation can be done using Newton's Second law and an application of the chain rule.

$$\vec{F} = m\vec{a}$$

Allowing the body force $\vec{F} = \vec{b}$ and substituting density for mass, we get a similar equation,

$$\vec{b} = \rho \frac{d}{dt} \vec{V}(x, y, z, t)$$

This can be done as we are operating with a fixed control volume and infinitesimal fluid parcels.

The body force $\vec{b}$ is a force that acts throughout the body of fluid (as opposed to a shear force which acts parallel to a plane).

Applying chain rule to the derivative of velocity,

$$\vec{b} = \rho \left(\frac{\partial \vec{V}}{\partial t} + \frac{\partial \vec{V}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \vec{V}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \vec{V}}{\partial z}\frac{\partial z}{\partial t} \right)$$

Equivalently, $\vec{b} = \rho \left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right)$

Substituting the value in the parentheses for the definition of material derivative, we obtain the final equation:

$$\boxed{\rho \frac{D\vec{V}}{Dt} = \vec{b}}$$

Equations of motion.

The conservation equations derived before in addition to a few assumptions about the forces and the behaviour of fluids lead to the equations of motion for fluids.
We assume that the body force on the fluid parcels is due to two components, fluid Stresses and other external forces.

$$\vec{b} = \nabla \cdot \sigma + \vec{f}$$

Here, σ is the Stress tensor and $\vec{f}$ represents external forces.
The fluid Stress is represented as the divergence of the stress tensor because the divergence ~~of the stress tensor~~ is the extent to which the tensor acts as a Source or sink; in other words, the divergence of the tensor results in a momentum Source or Sink also known as a force. For many applications, it is sufficient to say that $\vec{f}$ is composed only of gravity but the equation can be left in its most general form.

Cauchy stress Tensor.??

Stresses

The equations of motion depend on the Stress tensor σ. The tensor can be represented as

$$\sigma = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

A tensor is a generalisation of the concept of the higher-order quantity, a vector is represented as a first order tensor, a matrix a second order tensor a 3D matrix is a third order tensor and so on...

$\rho = \frac{m}{V}$

$\gamma = f/m$

$m = f/\gamma$

$\rho = \frac{f}{\gamma V}$

$f/V = \rho\gamma$

General form of the Navier-Stokes Equation.

The stress tensor σ is often divided into two terms. These are volumetric stress tensor which tends to change the volume of the body and the stress deviator tensor which tends to deform the body. The volumetric stress tensor represents the force which sets the volume of the body (the pressure forces). The stress deviator tensor represents the forces which determine body deformation and movement and is composed of the shear stresses on the fluid. Thus σ is broken down into

$$\sigma = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} = -\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} + \begin{pmatrix} \sigma_{xx}+p & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}+p & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz}+p \end{pmatrix}$$

Denoting the stress deviator tensor as T, we can make the substitution

$$\sigma = -p\underbrace{I}_{\text{Identity matrix.}} + T$$

Substituting into the previous equation, we obtain the most general form of the NS equation.

$$\boxed{\rho \frac{D\vec{V}}{Dt} = -\nabla p + \nabla \cdot T + \vec{f}}$$

Although this is the general form, it cannot be applied until it has been more specified. First off, depending on the type of fluid an expression must be determined for the stress tensor T; Secondly, if the fluid is not assumed to be incompressible, an equation of state and an equation dictating conservation of energy are necessary.

Convective acceleration.
Body force term.

Physical explanation of the Navier-Stokes equation.

- $\rho \underbrace{\frac{D\vec{v}}{Dt}}$ is the force on each fluid particle

  Local accleration

This equation states that the force is composed of 3 terms

* Turbulence ??

- $-\nabla p$: A pressure term (also known as volumetric stress tensor) which prevents motion due to normal stresses. The fluid presses against itself and keeps it from shrinking in volume.

- $\nabla \cdot T$: A stress term (known as the stress deviator tensor) which causes motion due to horizontal friction and shear stresses. The shear stress causes turbulence and viscous flows - If* one drags his hands through a liquid, the moving liquid will cause nearby liquid to start moving in the same direction.
  Turbulence is the result of the shear stress.

Investigate chemical nature.

- $\vec{f}$: The force which is acting on every fluid particle.

This intuitively explain turbulent flows and some common scenarios. For example, if water is sitting in a cup, the force (gravity) ρg is equal to the pressure term because $\frac{d}{dz}(\rho g z) = \rho g$. Thus since gravity is equivalent to the pressure, the fluid will sit still which is indeed what we observe when water is sitting in a cup.

Navier-Stokes fluids.

The general form of the NS equation cannot be used in practice as it has a number of unspecified elements. These are usually specific to the fluid which the equation is being applied to. A number of fluid models exist, varying the tensor T and the equation of state for (compressible fluids).

Describe →

Newtonian fluids

For simplicity of derivation, we will assume that the Newtonian fluid is incompressible, though in truth many fluids (such as air) are compressible, and the equations and methods have been thoroughly studied for compressible Newtonian fluids as well.

Fluid Stress

The basis for the Newtonian fluid is the assumption about the nature of the stress tensor. For a Newtonian fluid, the stress is proportional to the rate of deformation (the change in velocity in the direction of the stress). In other words,

$$T_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

The proportionality constant μ is called the viscosity of the fluid and defines how easily the fluid flows when subjected to body forces.

Stress divergence

The Navier-Stokes equation uses the divergence of stress $\nabla \cdot T$. We can calculate the stress term

$$\nabla \cdot \sigma = \mu \nabla \cdot \begin{pmatrix} \sigma_{xx} & T_{xy} & T_{xz} \\ T_{yx} & \sigma_{yy} & T_{yz} \\ T_{zx} & T_{zy} & \sigma_{zz} \end{pmatrix}$$

$$= \mu \begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial z} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial z} & 2\frac{\partial w}{\partial z} \end{pmatrix}$$

We can calculate the x-term of the divergence:

$$(\nabla\cdot\sigma)_i = \mu \frac{\partial}{\partial x}\left(2\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$= \mu \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}$$

$$= \mu \nabla^2 u + \mu \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$= \mu \nabla^2 u + \mu \frac{\partial}{\partial x}(\nabla \cdot v)$$

$$= \nabla^2 u + 0$$

$$= \mu \nabla^2 u$$

We can extend this to the other divergence terms, so we can replace the divergence with a vector laplacian:

$$\nabla\cdot T = \mu \nabla^2 v.$$

Newtonian fluid Equation

Without making any assumption about the form of the body force f, the final equation for an incompressible Newtonian fluid would be

$$\cancel{\rho \frac{Dv}{Dt} = -\nabla p + \mu \nabla^2 v + f}$$

$$\boxed{\rho \frac{Dv}{Dt} = -\nabla p + \mu \nabla^2 v + f}$$

Non - Newtonian fluids.

Though Newtonian fluids are the most common and the most studied of all fluid models, a few others exist as well. (eg. Ketchup - Becomes runnier when shaken)
↳ Honey, Custard, toothpaste, starch suspension, paint, blood and Shampoo.

Bingham Plastics

A Bingham plastic is similar to a Newtonian fluid but differs in one key aspect: While Newtonian fluids will start motion as soon as some shear stress is applied, a Bingham plastic fluid is able to resist stress until the stress passes a certain yield threshold. Afterwards, it will act the same as a Newtonian fluid. Some examples of Bingham plastics include toothpaste and clay.

Power Law fluids

Power-Law fluids differ from Newtonian fluids in that the stress in a Power-law fluid is raised to an exponent n, called the behaviour index. While Newtonian fluids follow the equation :

$$T = \mu \left(\frac{\partial u}{\partial y} \right)$$

power-Law fluids model stress according to the equation

$$T = \mu \left(\frac{\partial u}{\partial y} \right)^n$$

Chemistry ← of the fluids & Graph / behaviour

Thus, Newtonian fluids are a subset of Power-Law fluids since a Newtonian fluid is a Power-Law fluid with $n = 1$. When $n < 1$, the material is termed a pseudoplastic; pseudoplastics decrease in ~~the~~ viscosity as the shear stress increases. Common examples of pseudoplastics include lava, whipped cream, blood and many polymer solutions. If $n > 1$ the material is termed a dilant; dilants are rare, they have increasing viscosity with increasing shear stress. They tend to act in very non-intuitive manner.
For example ooblek (a mixture of cornstarch and water) become nearly solid when put under stress but remains liquid otherwise. (This would allow someone to run in a pool of ooblek but sink if they stood still in it).

Euler Equations.

Due to the complex nature of the Navier-Stokes equations, analytic solutions range from difficult to practically impossible, So the most common way to make use of the NS equation is through simulation and approximations (using computers).

From $\vec{b} = \nabla \cdot \sigma + \vec{f}$

and $\rho \frac{DV}{Dt} = -\nabla P + \mu \nabla^2 V + f$

diffusion ??

f is an external body force which act on every particle. $\vec{f}$ can be considered only due to gravity but can also be forces due to electric field and magnetic fields.

- A body force is a force which acts throughout the volume of a body.

For Fluids which are unaffected by electromagnetism (gravity only)

f can be considered due to gravity only.

Since volume is controlled and mass is conserved, density (ρ) can be used to replace mass. ①

Let γ be the body force per unit mass

$$\gamma = \frac{F_{body}}{m}$$

From statement ①

$$\gamma = \frac{F_{body}}{\rho}$$

$$\Rightarrow F_{body} = \gamma \rho$$

$$\Rightarrow \rho \frac{DV}{Dt} = -\nabla P + \mu \nabla^2 V + \gamma \rho$$

Breaking down the material $\frac{D\vec{v}}{Dt}$ yields,

$$\frac{DU}{Dt} = \frac{dU}{dt} + (\vec{v}\cdot\nabla)U$$

Since $\rho = m$

$\downarrow$

$F = ma$

$\downarrow$

$F = \rho a$

$\downarrow$

All terms have unit of force ??

⇒ Navier-Stokes Equation can be rewritten in a more specific form as

$$\boxed{\underbrace{\rho\frac{\partial\vec{v}}{\partial t}}_{\text{Local accleration}} + \underbrace{\rho(\vec{v}\cdot\nabla)\vec{v}}_{\text{Convective accleration}} = \underbrace{-\nabla P}_{\text{Pressure gradient}} + \underbrace{\gamma\rho}_{\text{Body force term}} + \underbrace{\mu\nabla^2\vec{v}}_{\text{Viscous term}}}$$

* Turbulence ??

Note : $(\vec{v}\cdot\nabla)\vec{v} = u\frac{\partial\vec{v}}{\partial x} + v\frac{\partial\vec{v}}{\partial y} + \omega\frac{\partial\vec{v}}{\partial z}$

Accleration

Total accleration is the sum of local accleration and convective accleration. The local accleration is the accleration of the fluid particles due to a unsteady flow. It is the change of velocity in a different time. The fluid particles may be acclerated even in a steady flow. *It can be due to a change of position of the fluid particle. This is due to advecting or convecting to a new location. So, it is called advective or convective accleration.

Flow	Local accleration	convective accleration	Example
• Steady & uniform flow	0	0	fluid flow at a constant rate through a uniform duct.
• unsteady & uniform flow	Non-zero	0	fluid flowing at a varying rate through a uniform duct
• Steady & non-uniform flow	0	Non-zero	fluid flow at constant rate through varying diameter duct.
• unsteady & non-uniform flow	Non-zero	Non-zero	fluid flowing at varying rate through varying diameter duct.

Kolmogorov Microscales

Andrey Nikolaevich Kolmogorov was a Russian mathematician who made significant contributions to the mathematics of probability theory and turbulence.

↓

Kolmogorov - Smirnov Test.

In the view of Kolmogorov, ~~turbulence~~ turbulent motions involve a wide range of scales. From macroscale at which energy is supplied to microscale at which energy is dissipated by viscosity.
For example, consider a cumulus cloud. The macroscale of the cloud can be of the order of kilometers and may grow or persist over long periods of time. Within the cloud, eddies may occur over the scales of the order of millimetres. For small flow such as in pipes, the microscales may be much smaller. Most of the kinetic energy of the turbulent flow is contained in the macroscale structures. The energy "cascades" from these macroscale structures to microscale structures by an inertial mechanism. This process is known as turbulent energy cascade.
The smallest scales in turbulent flow are known as kolmogorov microscales. These are small enough that molecular diffusion becomes important and viscous dissipation of energy takes place and the turbulent kinetic energy is dissipated ~~as~~ into heat.

The smallest scales in turbulent flow i.e. the kolmogorov microscales are:

Kolmogorov length scale $\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$

Kolmogorov time scale $\eta = \left(\frac{\nu}{\varepsilon}\right)^{1/2}$

Kolmogorov velocity scale $U_\eta = (\nu\varepsilon)^{1/4}$

Where ε is the average rate of dissipation of turbulence kinetic energy per unit mass and has dimensions (m^2/s^3). ν is the kinematic viscosity of the fluid and has dimensions (m^2/s)

$\mathbb{R}^2 \rightarrow \mathbb{R}^3$

$\mathbb{R} \rightarrow \mathbb{C}$

The Size of the smallest eddy in the flow is determined by Viscosity. The Kolmogorov length scale decreases as Viscosity decreases. For very high Reynold's Number flow, the viscous forces are smaller with respect to inertial forces. Smaller scale motions are then necessarily generated until the effects of viscosity become important and energy is dissipated. The ratio of the largest to smallest length scales in the turbulent flow are proportional to the Reynold's Number (Raised to $3/4$).

$$\frac{L}{\eta} \sim \left(\frac{UL}{\nu}\right)^{3/4} = Re^{3/4}$$

This causes a direct numerical simulations of turbulent flow to be practically impossible. For example, consider a flow with a Reynolds Number of 10^6. In this case, the ratio L/I is proportional to $10^{18/4}$. Since we have to analyse 3D problems, we need to compute a grid that consisted of at least 10^{14} points. This far exeeds the capacity and possibilities of existing computers.

LORENZ Equations from N·S·E

Boussinesq Approximation.

The Boussinesq approximation is applied to problems where the fluid varies in temperature from one place to another driving a flow of fluid and heat transfer.
The fluid satisfies conservation of mass/momentum/energy. In B·A, variations in fluid properties other than density ρ are ignored and ρ only appears when it is multiplied by g, (grav. acc)

If u is the local velocity of a parcel of fluid, the continuity equation for conservation of mass is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$$

If density variations are ignored, this reduces to

$$\nabla \cdot u = 0$$

General expression for cons. of momentum of an incompressible Newtonian fluid (N·S·E) is,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p + \underbrace{\nu}_{\text{Kinematic viscosity}} \nabla^2 u + \frac{1}{\rho} \underbrace{F}_{\Sigma \text{ of any body force terms (eg. gravity)}}$$

In this equation, density variations are assumed to have a fixed part and another part that has a linear dependence on temperature: $\rho = \rho_0 - \underbrace{\alpha}_{\text{coeff. of thermal expansion.}} \rho_0 \Delta T$

If $F = \rho g$, the resulting conservation equation is

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u - g \alpha \Delta T$$

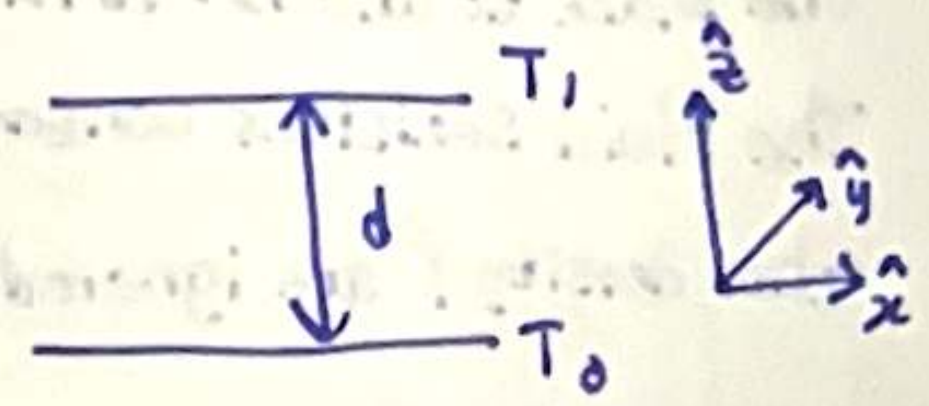
In the eqn for heat flow in a temperature gradient, the heat capacity per unit volume, ρC_p is assumed constant. The resulting eqn is:

$$\frac{\partial T}{\partial t} + u \cdot \nabla T = K \nabla^2 T + \frac{J}{\rho C_p}$$

J: rate per unit volume of internal heat production
K: thermal diffusivity.

Considr the incompressible NSEs within the Boussinesq approximation, modelling the thermal convection between 2 parallel horizontal flat plates separated by a distance d and kept at temperatures T_0 (bottom plate) and T_1 (top plate) (with $T_1 < T_0$):

Boussinesq approximation
$$\begin{cases} \nabla \cdot u = 0 \\ \dfrac{\partial u}{\partial t} + u \cdot \nabla u = -\dfrac{1}{\rho_0} \nabla p + \nu \nabla^2 u + \alpha g (T - T_0)\hat{z} \\ \dfrac{\partial T}{\partial t} + u \cdot \nabla T = \kappa \nabla^2 T \end{cases}$$



Here: ρ_0 is density of fluid at temperature T_0, ν is the kinematic viscosity of the fluid, α is the coeff. of thermal expansion, κ is the thermal diffusivity and $\hat{z}$ is a unit vector in the vertical direction.

Dimensionless variables are introduced (simpler Maths and does not affect flow)

$\downarrow$ Non-dimensionalisation

Dimensionless temperature disturbance is also introduced.

$$x^* = \frac{x}{d} \qquad u^* = \frac{d}{\kappa} u \qquad t^* = \frac{\kappa}{d^2} t \qquad p^* = \frac{1}{\rho_0}\left(\frac{d}{\kappa}\right)^2 p \qquad \Bigg| \qquad \theta^* = \frac{T - T_0}{T_1 - T_0} - z^*$$

where z^* is the dimensionless vertical coordinate.

Governing Eqns become

$$\nabla \cdot u = 0 \qquad \text{(a)}$$

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + Pr \nabla^2 u + Pr\, Ra\, \theta \hat{z}$$

$\longrightarrow$ Asterisks (*) for dimensionless variables are omitted.

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = w + \nabla^2 \theta$$

$\hookrightarrow$ Pr = Prandtl number $= \nu / \kappa$
$\hookrightarrow$ material property

Ra = Rayleigh number
$\hookrightarrow$ $Ra = \dfrac{\alpha g (T_0 - T_1) d^3}{\nu \kappa}$

u, v, w are velocity vectors in x, y, z directions respectively

We assume that the boundaries at $z=0$ and $z=1$ are stress free, rigid and isothermal, leading the boundary condition

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial^4 w}{\partial z^4} = 0 \quad \text{and } \theta = 0 \text{ at } z=0 \text{ and } z=1$$

2D - Convection rolls.

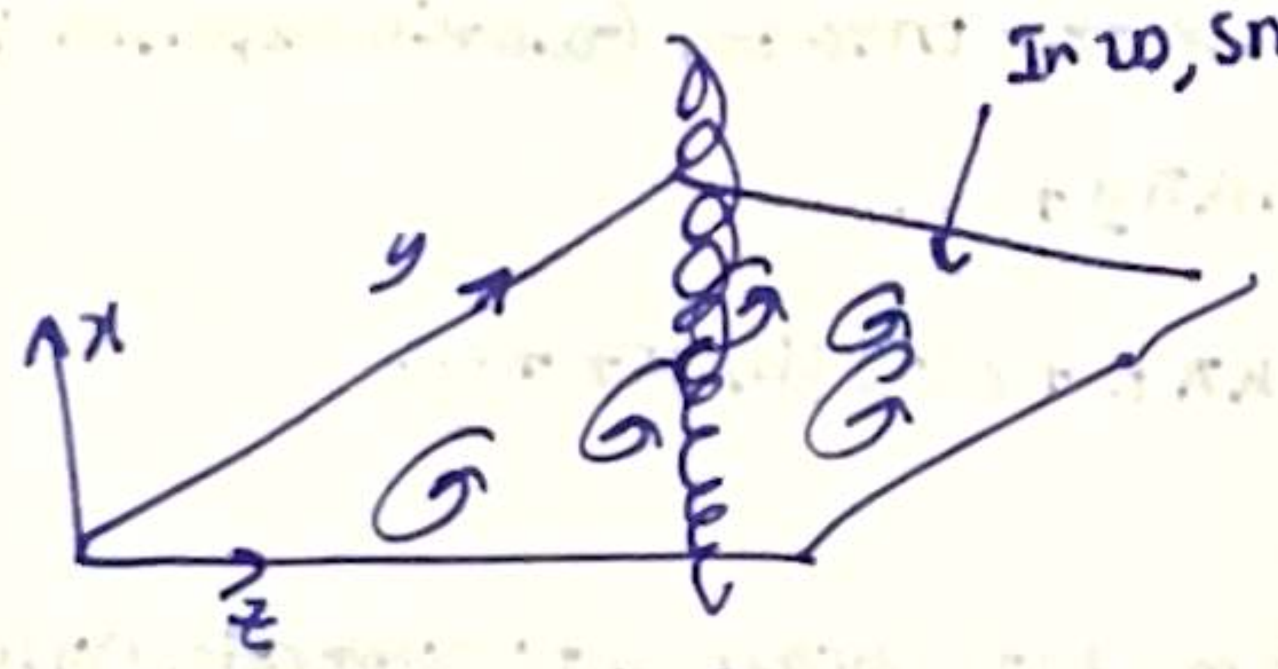
We now consider the convection in the form of two-dimensional convection rolls, say aligned with the x-axis. In this case, $u=0$ and v, w and θ only depend on y, z and t. We introduce the two dimensional streamfunction $\psi(y,z,t)$ such that

$$v = \frac{\partial \psi}{\partial z} \quad \text{and } w = -\frac{\partial \psi}{\partial y}$$

in which case the continuity equation $\nabla \cdot u = 0$ is automatically satisfied.

In 2D, the vorticity only has one non zero component ξ in the x direction:

In 2D, small eddies add up to form a large vortex.



$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial x} = -\nabla_2^2 \psi \quad \text{where } \nabla_2^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

By taking the curl of the momentum, eqn (a) and projecting in the x-direction, we can eliminate pressure p from the equations and obtain the following for ξ

$$\frac{\partial \xi}{\partial t} + u \cdot \nabla \xi = Pr \nabla_2^2 \xi + Pr Ra \frac{\partial \theta}{\partial y}$$

where $J(\chi, \psi)$ is the jacobian of (χ, ψ) w.r.t (y, z)

Also, it is seen that for any scalar quantity χ we have

$$u \cdot \nabla \chi = v \frac{\partial \chi}{\partial y} + w \frac{\partial \chi}{\partial z} = \frac{\partial \psi}{\partial z}\frac{\partial \chi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \chi}{\partial z} = J(\chi, \psi)$$

Using these observations, the governing eq$\underline{n}$s for the 2D convection rolls becomes:

$$\frac{\partial \xi}{\partial t} + J(\xi, \psi) = Pr\, \nabla_2^2 \xi + Pr\, Ra \frac{\partial \theta}{\partial y}$$

$$\frac{\partial \theta}{\partial t} + J(\theta, \psi) = \nabla_2^2 \theta - \frac{\partial \psi}{\partial y}$$

$$\xi = -\nabla_2^2 \psi$$

<u>Truncated Galerkin Expansion</u>

Galerkin method - Convert continuous operator problem (such as differential equation) to a discrete problem.

We now proceed to reduce the above set of PDEs to ODEs which will lead to the Lorenz equations.

The simplest approach is to consider a severely truncated Galerkin expansion:

$$\psi(y,z,t) = a(t) \sin \pi z \sin k\pi y + \ldots$$

$$\theta(y,z,t) = b(t) \sin \pi z \cos k\pi z + c(t) \sin 2\pi z + \ldots$$

Where the 2 terms involving $a(t)$ and $b(t)$ correspond to convection rolls with wave n$\underline{o}$ k in the y-direction and the term involving $c(t)$ can be justified by considering the modification of the mean temperature profile due to convection. We thus have:

$$\zeta(y,z,t) = \pi^2 (1+k^2)\, a(t) \sin \pi z \sin k\pi z + \ldots$$

$$\frac{\partial \zeta}{\partial y}(y,z,t) = k\pi^3 (1+k^2)\, a(t) \sin \pi z \cos k\pi y + \ldots$$

$$\frac{\partial \zeta}{\partial z}(y,z,t) = \pi^3 (1+k^2)\, a(t) \cos \pi z \sin k\pi y + \ldots$$

$$\frac{\partial \psi}{\partial y}(y,z,t) = k\pi\, a(t) \sin \pi z \cos k\pi y + \ldots$$

$$\frac{\partial \psi}{\partial z}(y,z,t) = \pi\, a(t) \cos \pi z \sin k\pi y + \ldots$$

$$\frac{\partial \theta}{\partial y}(y,z,t) = -k\pi\, b(t) \sin \pi z \sin k\pi y + \ldots$$

$$\frac{\partial \theta}{\partial z}(y,z,t) = \pi\, b(t) \cos \pi z \cos k\pi y + 2\pi\, c(t) \cos 2\pi z + \ldots$$

The Jaccobinan is defined as

$$J(f,g) = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix} = \frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}$$

We also obtain :

$$J(\xi,\psi) = \frac{1}{4} k\pi^4(1+k^2)a^2(t)\left[\sin 2\pi z \sin 2k\pi y - \sin 2\pi z \cos 2k\pi y\right] + \cdots$$

$\cos k\pi y + \cdots$

$$J(\theta,\psi) = -\frac{1}{2} k\pi^2 a(t)b(t)\left[\sin 2\pi z \sin^2 k\pi y + \sin 2\pi z \cos^2 k\pi y\right] - 2k\pi^2 a(t)c(t)\sin\pi z \cos 2\pi z$$

In a Galerkin ~~expansion~~ method, the residual error is minimised by ensuring that the projection of the error on the basis functions retained is zero. The one-term Galerkin expansion of ξ is obtained by multiplying the ξ equation by $\sin\pi z \sin k\pi y$ and integrating z over $[0,1]$ and integrating y over $(0, 2/k)$ and gives after simplifications :

$$\frac{da}{dt} = -Pr\,\pi^2(1+k^2)a(t) - \frac{k\pi}{\pi^2(1+k^2)} Pr\,R\,a\,b(t)$$

Similarly the 2-term Galerkin expansion of θ gives :

$$\frac{db}{dt} + k\pi^2 a(t)c(t) = -\pi^2(1+k^2)b(t) - k\pi a(t)$$

$$\frac{dc}{dt} - \frac{1}{2}k\pi^2 a(t)b(t) = -4\pi^2 c(t)$$

Now rescaled set of Lorenz equations.

Rescaling

We rescale the equations as follows. Define:

$$J = \pi^2(1+k^2)t \qquad \hat{b}(J) = \frac{-k\,Ra\,b(t)}{\pi^3(1+k^2)^2} \qquad \hat{c}(J) = -\frac{k^2 Ra\, c(t)}{\pi^3(1+k^2)^3}$$

Which gives :

$$\frac{da}{dJ} = -\sigma a + \sigma \hat{b} \quad \text{where } \sigma = Pr$$

$$\frac{d\hat{b}}{dJ} = -\hat{b} + ra - a\hat{c} \quad \text{where } r = \frac{k^2 Ra}{\pi^4(1+k^2)^3}$$

$$\frac{d\hat{c}}{dJ} = -s\hat{c} + \frac{k^2}{2(1+k^2)^2}a\hat{b} \quad \text{where } s = \frac{4}{(1+k^2)}$$

Re scaling further as :

$$X = \frac{k}{\sqrt{2}(1+k^2)}a(J) \qquad Y = \frac{k}{\sqrt{2}(1+k^2)}\hat{b}(J) \qquad Z = \hat{c}$$

Gives the Standard form of the Lorenz equations :

$$\boxed{\begin{aligned} \frac{dX}{dJ} &= -\sigma X + \sigma Y \\ \frac{dY}{dJ} &= -Y + rX - XZ \\ \frac{dZ}{dt} &= -sZ + XY \end{aligned}}$$