Trigonometric Integrals

$$\int t \sin nt \, dt = \frac{1}{n^2}(\sin nt - nt\cos nt) + K$$

$$\int t \cos nt \, dt = \frac{1}{n^2}(\cos nt + nt\sin nt) + K$$

Series

- An Arithmetic progression is a sequence of numbers with a common difference.
- A geometric progression is a sequence of numbers with a common ratio.

- A Series is ~~Sequence of~~ Sum of a Sequence of numbers

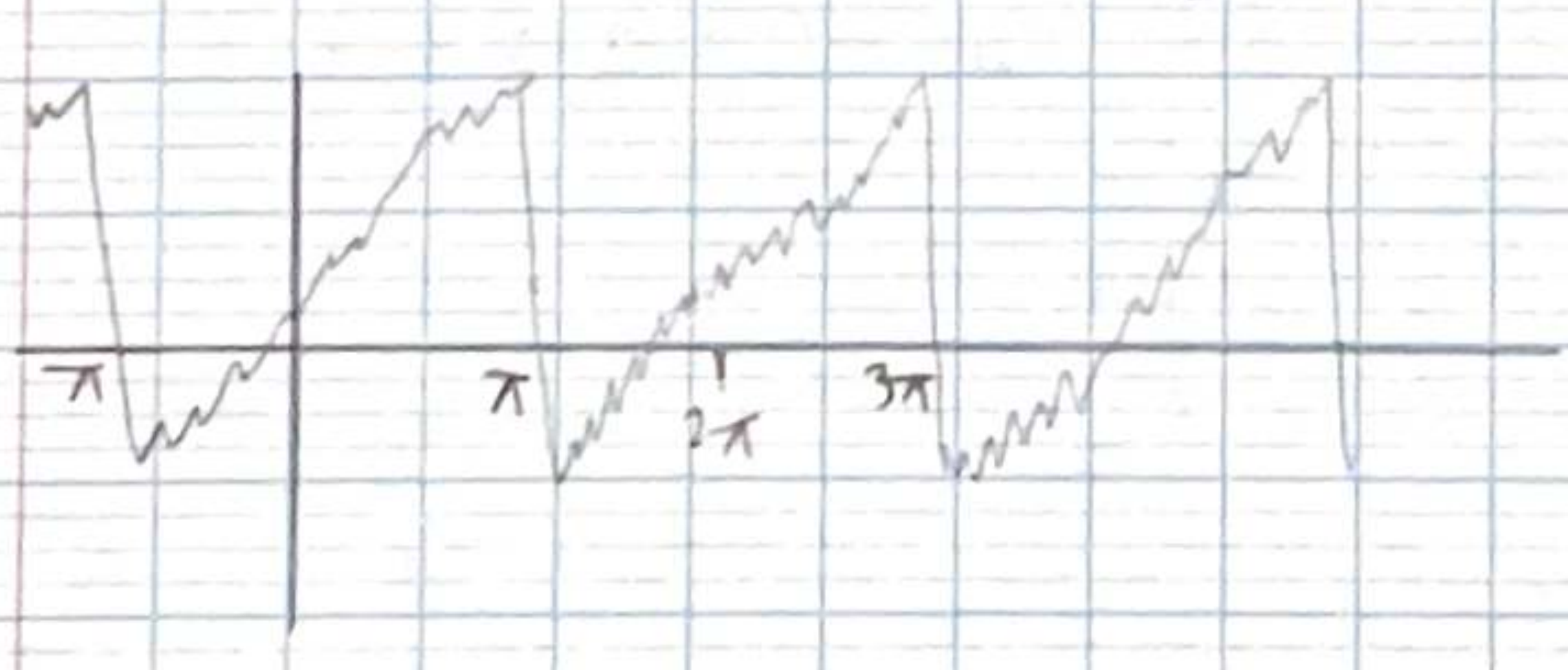
→ An infinite series that converges to a particular value has common ratio less than 1

eg. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots = \frac{3}{2}$ ←

↘ Maclaurin & taylor Series are used to express Certain functions as polynomials with infinite terms.

Fourier Series as a trigonometric infinite Series.

$$f(t) = 1 + 2\sin t - \sin 2t + \tfrac{2}{3}\sin 3t - \tfrac{1}{2}\sin 4t + \tfrac{2}{5}\sin 5t + \dots$$



The infinite fourier Series Converges to the Sawtooth Curve.

Full range Fourier Series.

A periodic waveform $f(t)$ of period $p = 2L$ has fourier Series given by:

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)}$$

$$= \frac{a_0}{2} + a_1 \cos\left(\frac{\pi t}{L}\right) + a_2 \cos\left(\frac{2\pi t}{L}\right) + a_3 \cos\left(\frac{3\pi t}{L}\right) + \dots$$

$$+ b_1 \sin\left(\frac{\pi t}{L}\right) + b_2 \sin\left(\frac{2\pi t}{L}\right) + b_3 \sin\left(\frac{3\pi t}{L}\right) + \dots$$

where a_n and b_n are the fourier Coefficients
and $\frac{a_0}{2}$ is the mean value (refered as the dc level)

Fourier coefficients for any range from $-L$ to L

If $f(t)$ is expanded in the range $-L$ to L (period $=2L$) so that the range of integration is $2L$, then the fourier coefficients are given by:

$$\boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt}$$

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt}$$

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt}$$

where $n = 1, 2, 3, \dots$

Dirichlet Conditions.

Any periodic waveform of period $p = 2L$ can be expressed in a fourier Series provided that:

(a) It has a finite number of discontinuities within the period $2L$.

(b) It has a finite average value in the period $2L$.

(c) It has a finite number of positive and negative minima and maxima.

When these conditions are satisfied, the fourier Series for the function $f(t)$ exists.

When period = 2L = 2π

If a function is defined in the range $-\pi$ to π, the range of integration is 2π.

In this case, fourier coefficients of the fourier series f(t) become:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

Formula for fourier series become: $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$

where $n = 1, 2, 3, \ldots$

Sketch the waveform of the periodic function defined as.

$f(t) = t$ if $-\pi < t < \pi$ — ①

$f(t) = f(t + 2\pi)\ \forall_t$

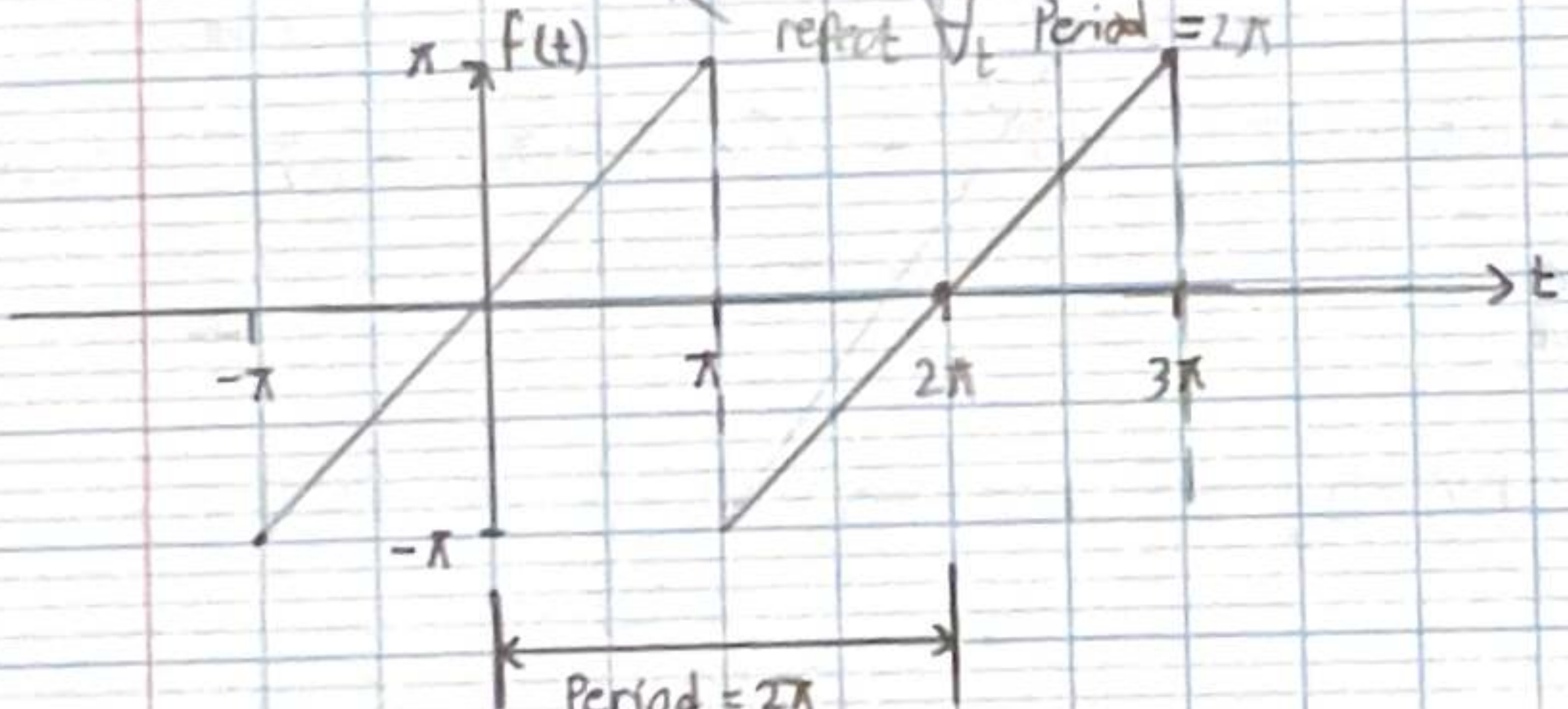
$y = x$ for $-\pi < x < 2\pi$

$$f(t) = \begin{cases} t \text{ if } -\pi < t < \pi \\ (t + 2\pi)\ \forall_t \end{cases}$$

t	$-\pi$	π
f(t)	$-\pi$	π

①

repeat $\forall_t$ Period $= 2\pi$



$2L = 2\pi \quad L = \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t\,dt$$

$$= \frac{1}{\pi}\left[\frac{t^2}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t \, \cos\frac{n\pi t}{L} \, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[t \cos\frac{n\pi t}{\pi}\right] dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt \, dt$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}(\cos nt + nt\sin nt)\Big|_{-\pi}^{\pi}\right.$$

$$\cancel{= \frac{1}{\pi}\left[\frac{1}{n^2}\right.}$$

$$= \frac{1}{\pi n^2}\left[\left[\cos n\pi + n\pi \underbrace{\sin n\pi}_{0}\right] - \left[\cos(-n\pi) - n\pi \underbrace{\sin(-n\pi)}_{+0}\right]\right]$$

$$\sin(-n\pi) = -\overbrace{\sin n\pi}^{=0}$$

$$= \frac{1}{\pi n^2}\left[(\cos n\pi + 0) - (\cos(-n\pi) + 0)\right]$$

$$\cos(-n\pi) = \cos n\pi$$

$= \cancel{\cos}$

$$\frac{1}{\pi n^2}(\cos n\pi - \cos n\pi)$$

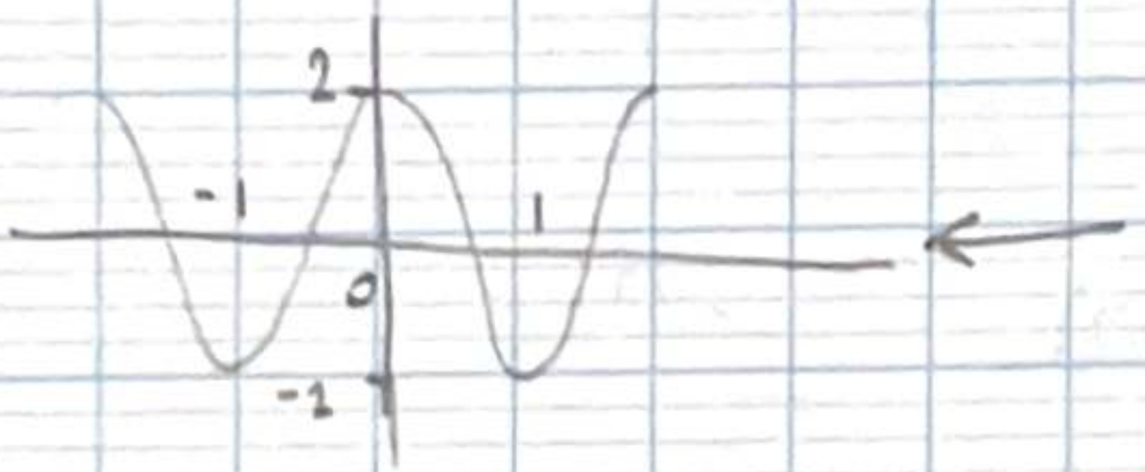
$$= 0 \leftarrow$$

Fourier Series of even & odd functions

Even functions

A function $y = f(t)$ is said to be even if $f(-t) = f(t)\ \forall_t$ (for all values of t).
The graph of an even function is always symmetrical about the y-axis. (it is a mirror image).

eg. $f(t) = 2\cos \pi t$



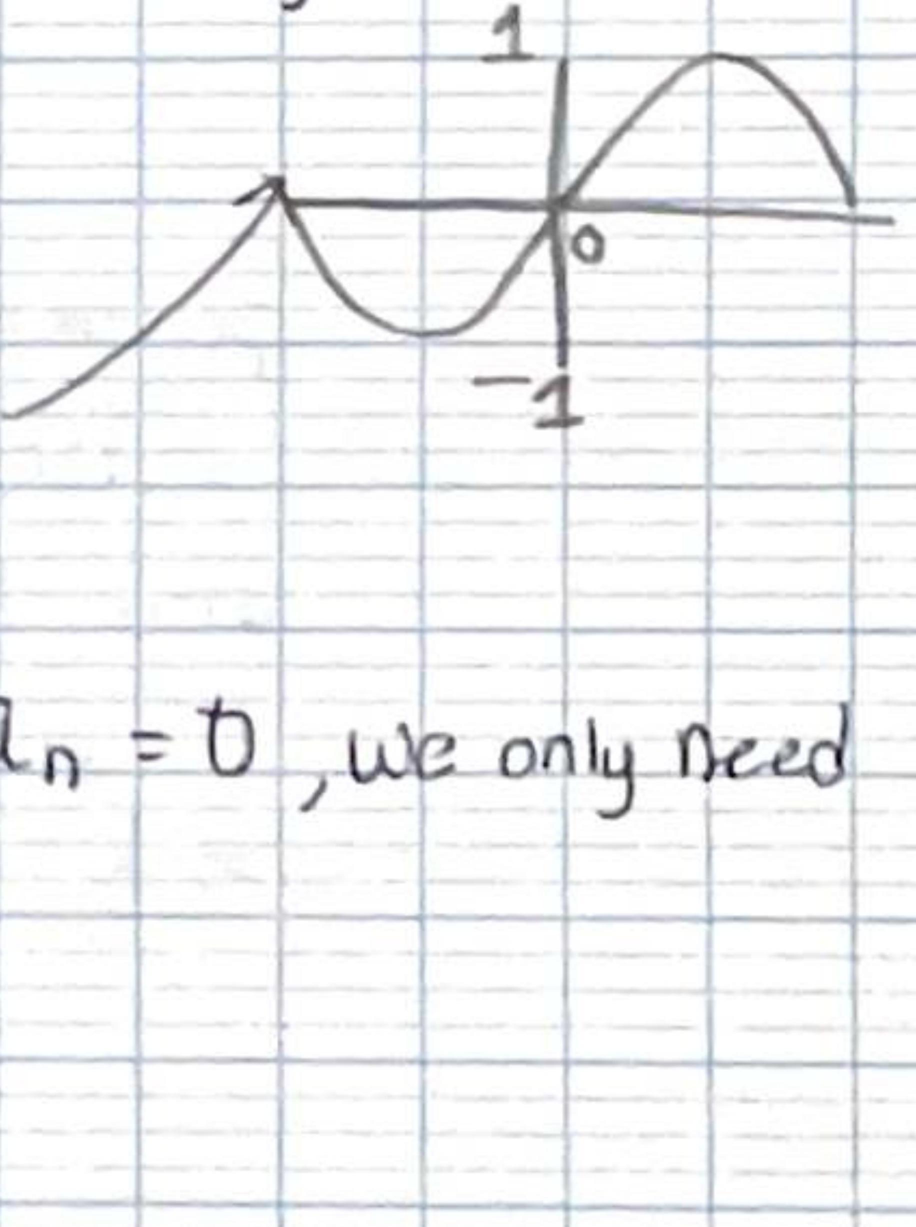
Fourier Series for even functions.

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt$$ and $f(t)$ is even, it means that this integral will have a value of 0 and we only need to calculate a_0 and a_n.

An even function has only cosine terms in its fourier expansion.

$$\text{for } f(t) \text{ even,}\quad f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi t}{L}$$

eg. $f(t) = \sin(t)$



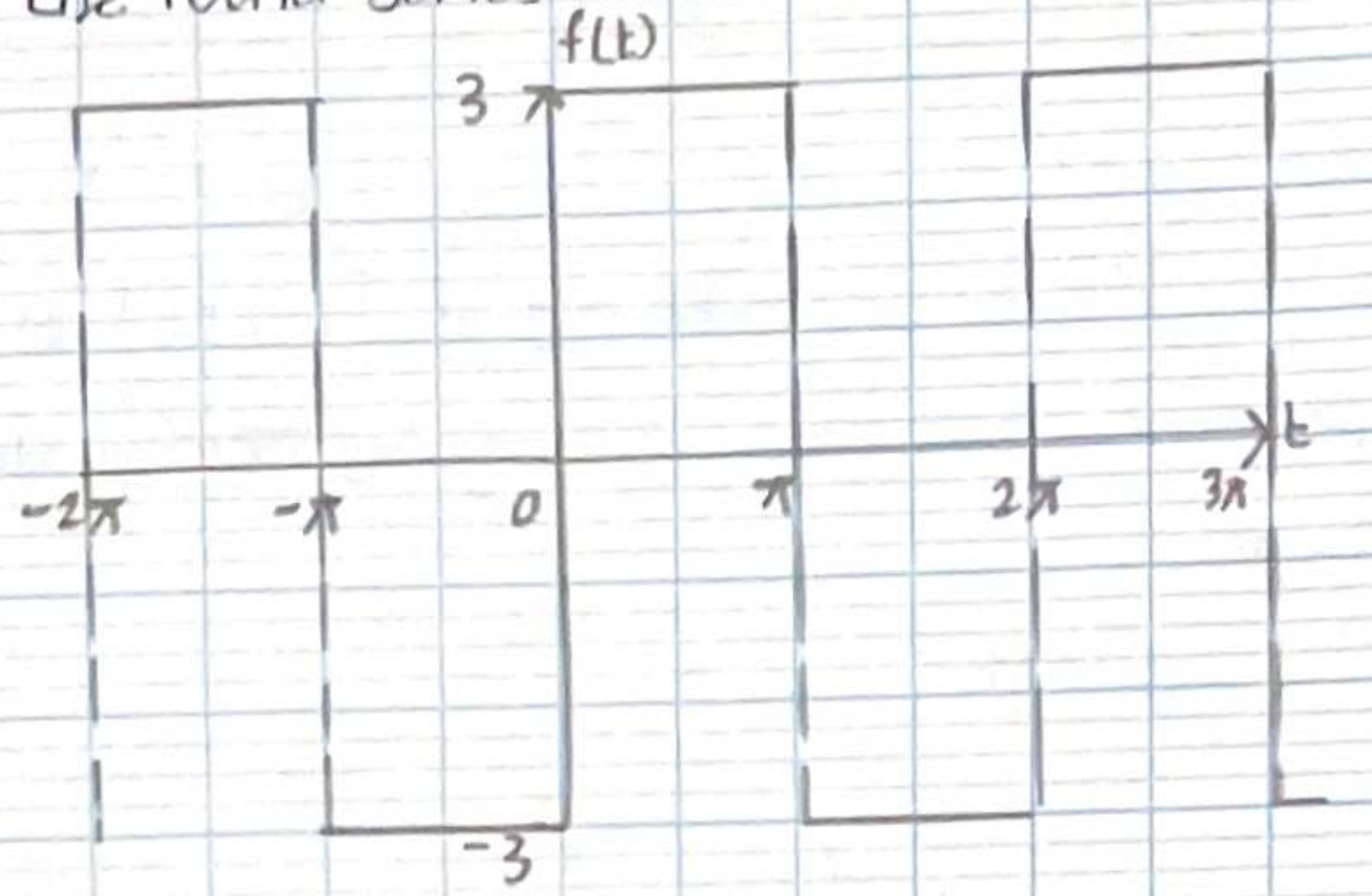
Odd functions → Opposite polarity about y-axis.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt \Rightarrow a_0 = 0 \ \&\ a_n = 0,$$ we only need to evaluate b_n

$$\text{for } f(t) \text{ odd,}\quad f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

An odd function has only Sine in its fourier expansion.

eg. Find the fourier Series for the function for which the graph is given by:



By inspection, f(t) is odd,

$\Rightarrow a_0 \ \& \ a_n = 0$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin \frac{n\pi t}{L} \, dt.$$

from graph,

$$f(t) = \begin{cases} -3 & \text{for } -\pi < t < 0 \\ 3 & \text{for } 0 < t < \pi \end{cases}$$

period is 2π

$\Rightarrow f(t) = f(t + 2\pi)$

also, $L = \pi$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt \, dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -3\sin nt \, dt + \int_{0}^{\pi} 3\sin nt \, dt\right]$$

$$= \frac{3}{\pi}\left[\int_{-\pi}^{0} -\sin nt \, dt + \int_{0}^{\pi} \sin nt \, dt\right]$$

$$= \cancel{3\pi}\ \frac{3}{\pi}\left[\left(\frac{\cos nt}{n}\Big|_{-\pi}^{0}\right) + \left(-\frac{\cos nt}{n}\Big|_{0}^{\pi}\right)\right]$$

$$\frac{3}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0\right]$$

$$= \frac{3}{n\pi}\left(2 - \cos \pi n - \cos n\pi\right)$$

$$= \frac{6}{n\pi}\left(1 - \cos \pi n\right)$$

$$= \frac{12}{n\pi} \ (n \text{ odd})$$

or $= 0 \ (n \text{ even})$

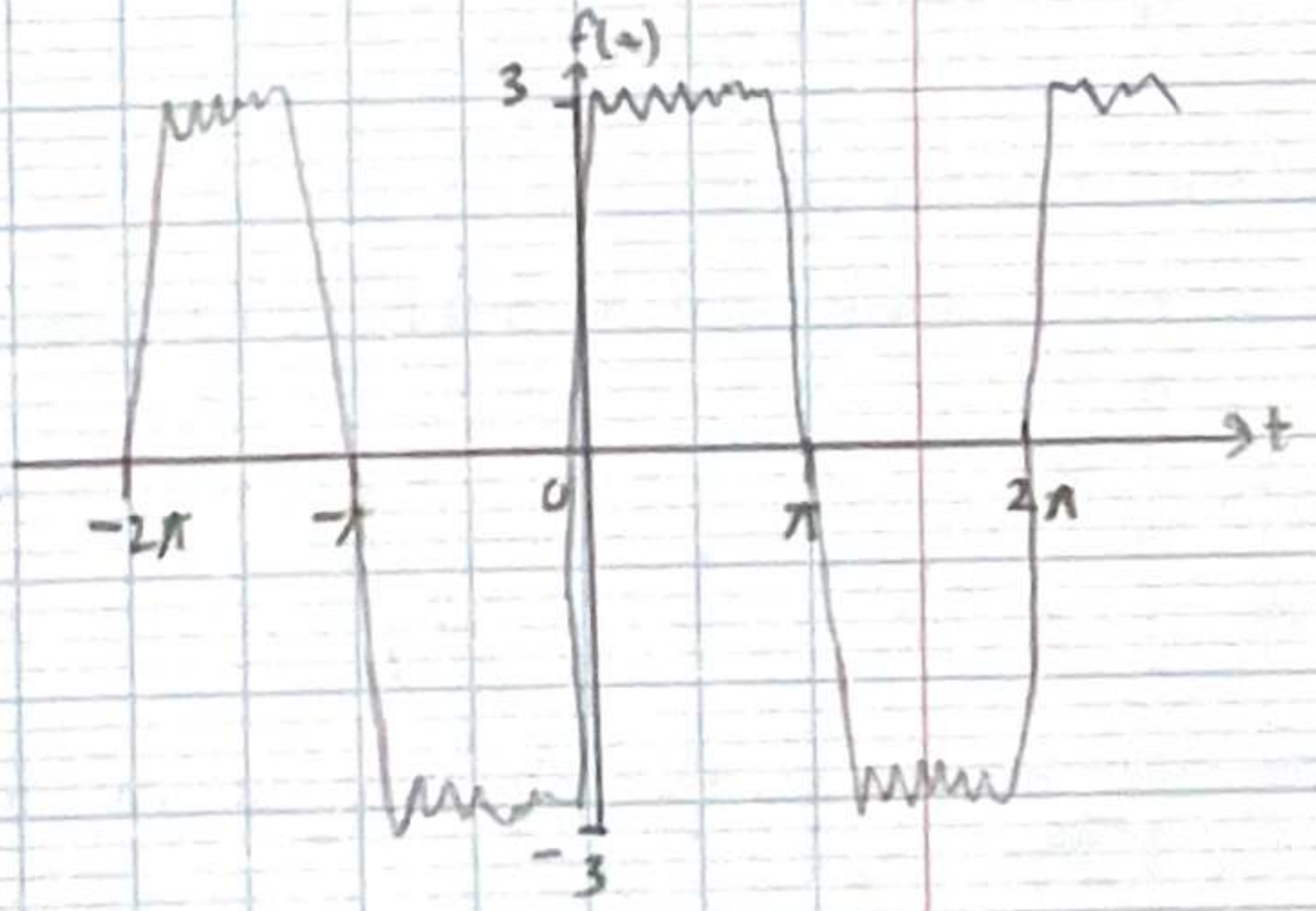
$$b_n = \frac{12}{\pi(2n-1)} \quad n, = 1, 2, 3, \ldots$$

$$f(t) = \sum_{n=1}^{\infty} \frac{12}{\pi(2n-1)} \sin(2n-1)t$$

$$= \frac{12}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)t}{(2n-1)} \rightarrow \frac{12}{\pi}\left(\sum_{n=1}^{\infty} \frac{\sin(2n-1)t}{(2n-1)}\right)$$

n	$\sum_{n=1}^{\infty} \frac{\sin(2n-1)t}{(2n-1)}$
1	$\sin t$
2	$\frac{1}{3}\sin 3t$
3	$\frac{1}{5}\sin 5t$
4	$\frac{1}{7}\sin 7t$
5	$\frac{1}{9}\sin 9t$

$$\Rightarrow \frac{12}{\pi}\left(\sin t + \tfrac{1}{3}\sin 3t + \tfrac{1}{5}\sin 5t + \tfrac{1}{7}\sin 7t + \tfrac{1}{9}\sin 9t + \ldots\right)$$



Fourier Series approximation to f(t) (square wave)

Half-range Fourier Series

If a function is defined over 0 to L instead of -L to L, it may be expanded in terms of sines only or cosines only to produce a half-range Fourier series.

Even functions and half-range Cosine Series

An even function can be expanded from half its range ie: 0 to L
-L to 0
L to 2L

The range of integration is L.

The Fourier Series of the half-range even function is given by:

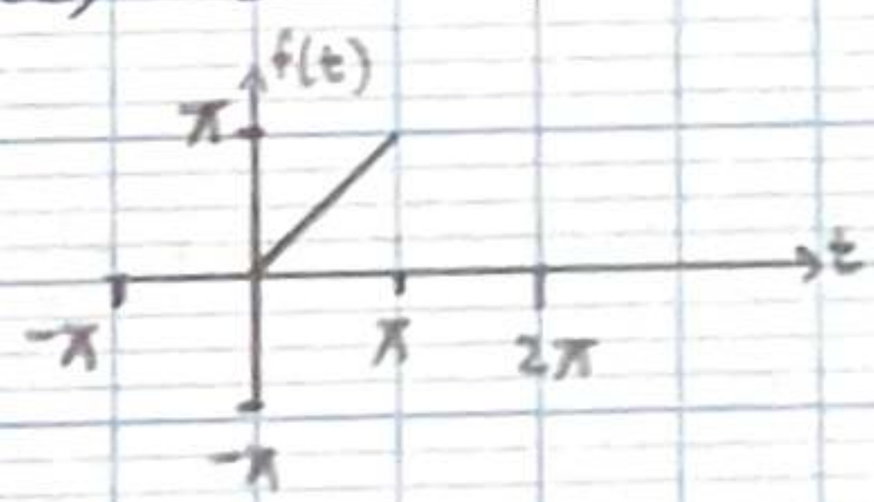
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi t}{L} \quad \text{for } n = 1, 2, 3, \text{ where}$$

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt$$

$$a_n = \frac{2}{L}\int_0^L f(t) \frac{\cos n\pi t}{L}\,dt$$

$$b_n = 0$$

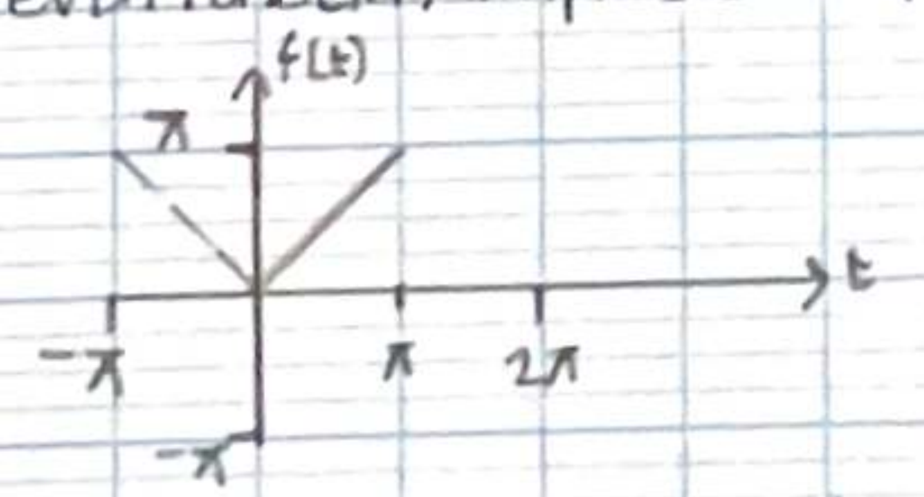
$f(t) = t$ is sketched from $t=0$ to $t=\pi$



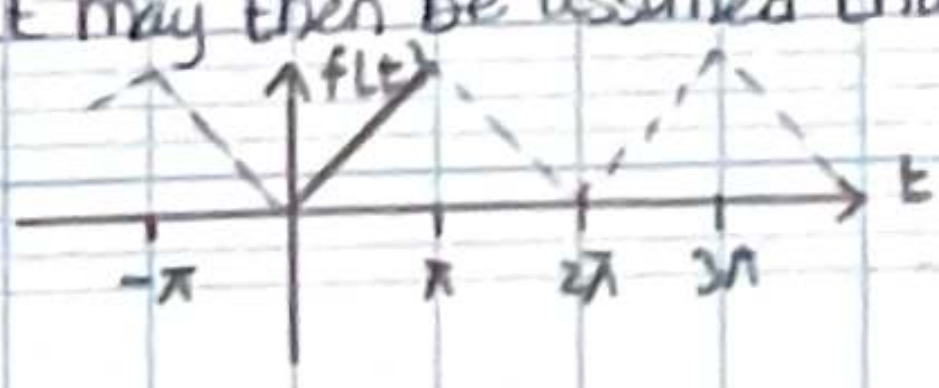
$f(t) = |t|$??

Since function is even, it is

An even function implies that it must be symmetrical about f(t) axis



It may then be assumed that the 'triangular wave form' produced is periodic with period = 2π



$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t \sin\frac{n\pi t}{\pi}\, dt$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}(\sin nt - nt\cos nt)\right]_{-\pi}^{\pi}$$

$$\frac{1}{n^2\pi}\left[\sin nt - nt\cos nt\right]_{-\pi}^{\pi}$$

$$= \frac{1}{n^2\pi}\left[(\underbrace{\sin n\pi}_{0} - n\pi\cos n\pi) - (\sin(-n\pi) + n\pi\cos(-n\pi))\right]$$

$$= \frac{1}{n^2\pi}\left[(0 - n\pi\cos n\pi) - (0 + n\pi\overbrace{\cos(-n\pi)}^{=\cos n\pi})\right]$$

$$= \frac{1}{n^2\pi}\left(-n\pi\cos n\pi - n\pi\cos(-n\pi)\right)$$

$$= \frac{1}{n^2\pi}\left(-2n\pi\cos n\pi\right)$$

$$= -\frac{2}{n}\cos n\pi$$

$$= -\frac{2}{n}(-1)^n$$

$$= \frac{2}{n}(-1)^{n+1}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nt + b_n\sin nt)$$

$$= \frac{0}{2} + \sum_{n=1}^{\infty}\left((0)\cos nt + \frac{2}{n}(-1)^{n+1}\sin nt\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{2}{n}(-1)^{n+1}\sin nt\right)$$

$$\rightarrow f(t) = 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t + \dots$$

n	$f(t)$
1	$2\sin t$
2	$-\sin 2t$
3	$\frac{2}{3}\sin 3t$
4	$-\frac{1}{2}\sin 4t$

Expressing fourier Series in angular displacement, ω form.

Let function $f(t)$ be periodic with period $T = 2L$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2L} \Rightarrow \omega = \frac{\pi}{L}$$

In this case, lower limit of integration $= 0$

$$\rightarrow f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

Note: half range of integration is $\frac{\pi}{\omega}$ $\Rightarrow L = \frac{\pi}{\omega}$

$$a_0 = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(t)\,dt \qquad a_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t) \cos n\omega t\,dt \qquad b_n = \int_0^{\frac{2\pi}{\omega}} f(t) \sin n\omega t\,dt$$

ω is measured in radians.

Let function $f(\omega)$ be periodic with period $2L$

$\theta = \omega t$

$$\Rightarrow f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \frac{\cos(n\pi\theta)}{L} + b_n \frac{\sin(n\pi\theta)}{L}\right)$$

Where

$$a_0 = \frac{1}{L}\int_0^{2L} f(\theta)\,d\theta$$

$$a_n = \frac{1}{L}\int_0^{2L} f(\theta) \cos \frac{n\pi\theta}{L}\,d\theta$$

$$b_n = \frac{1}{L}\int_0^{2L} f(\theta) \sin \frac{n\pi\theta}{L}\,d\theta$$

Harmonic Analysis

For full range fourier Series.

$$f(t) = \frac{a_0}{2} + \left(\sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt\right)$$

$$f(t) = \frac{a_0}{2} + \underbrace{(a_1 \cos t + b_1 \sin t)}_{\text{fundamental}} + \underbrace{(a_2 \cos 2t + b_2 \sin 2t)}_{\text{Second harmonic}} + \underbrace{(a_3 \cos 3t + b_3 \sin 3t)}_{\text{third harmonic}} + \dots$$

Odd harmonics

The fourier series will contain odd harmonics if $f(t+\pi) = -f(t)$

graphical example:

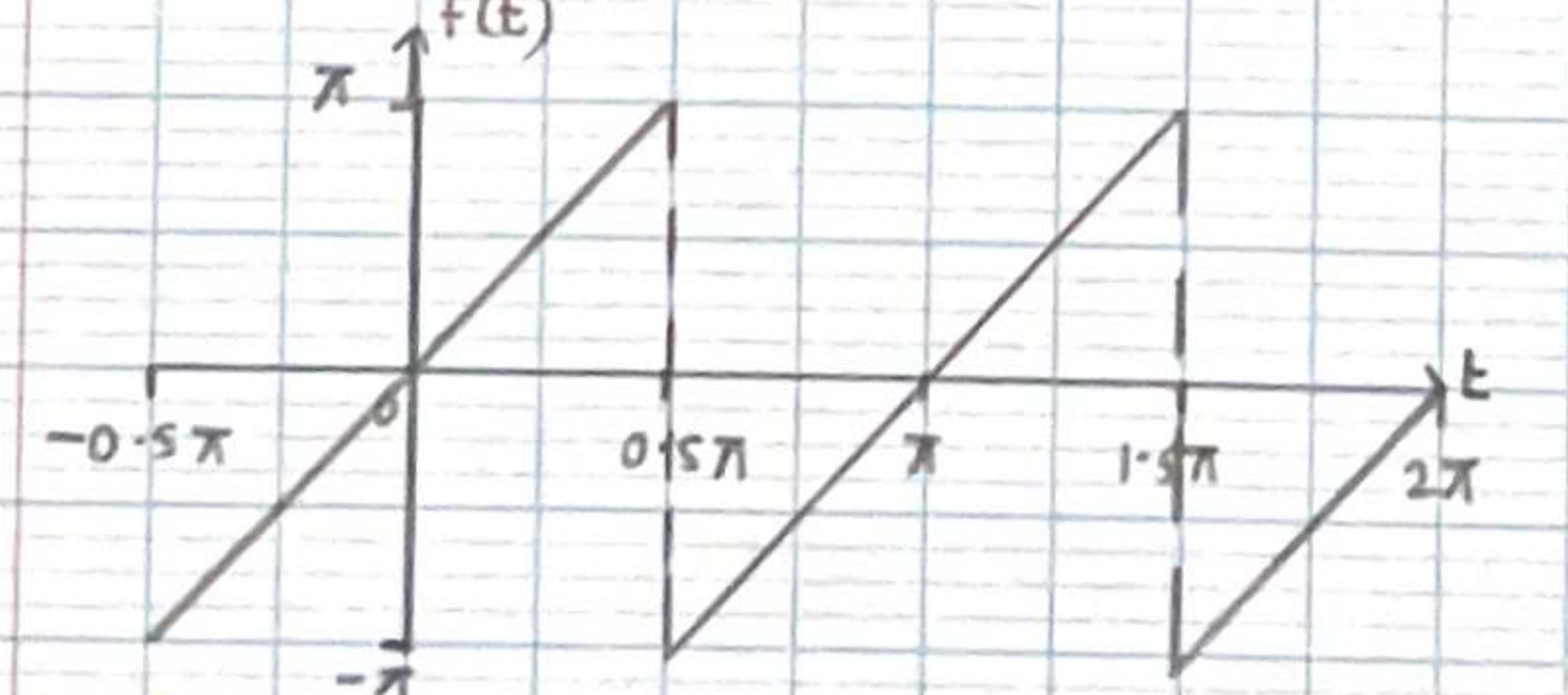
In this case, expansion will be.

$$f(t) = \frac{a_0}{2} + (a_1 \cos t + b_1 \sin t) + (a_3 \cos 3t + b_3 \sin 3t) + (a_5 \cos 5t + b_5 \sin 5t) + \dots$$

All the harmonics are odd.

Even harmonics

The fourier series will contain even harmonics if $f(t+\pi) = f(t)$, ie. if it has a period of π.



The fourier expansion will be.

$$f(t) = \frac{a_0}{2} + (a_2 \cos 2t + b_2 \sin 2t) + (a_4 \cos 4t + b_4 \sin 4t) + (a_6 \cos 6t + b_6 \sin 6t) + \dots$$

All the harmonics are even.

Determine the existence of odd or even harmonics for

- $f(t) = \begin{cases} -t - \frac{\pi}{2} & -\pi \leq t < 0 \quad -(1) \\ t - \frac{\pi}{2} & 0 \leq t < \pi \quad -(2) \end{cases}$

- $f(t) = f(t + 2\pi)$ ← period $= 2\pi$

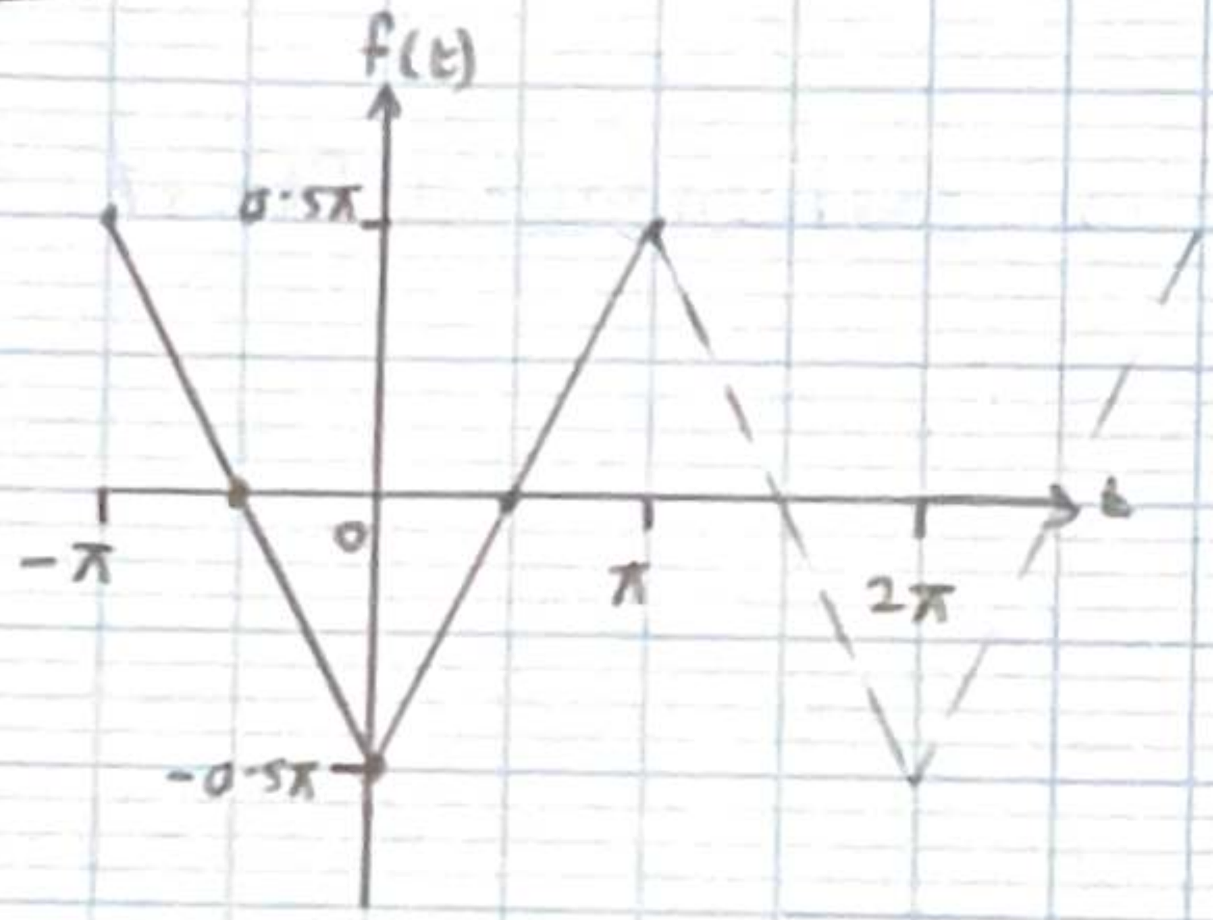
$f(t) = -t - \frac{\pi}{2} \quad -(1)$

$f(-\pi) = \pi - \pi/2 = \pi/2 = 0.5\pi$

$f(-\pi/2) = \pi/2 - \pi/2 = 0$

$f(0) = 0 - \pi/2 = -\pi/2 = -0.5\pi$

Graph



even graph with odd harmonics.

$f(t) = t - \pi/2 \quad -(2)$

$f(0) = 0 - \pi/2 = -\pi/2 = -0.5\pi$ ← Same as (1)

$f(\pi/2) = \pi/2 - \pi/2 = 0$

$f(\pi) = \pi - \pi/2 = \pi/2 = 0.5\pi$

from graph, $f(t+\pi) = -f(t)$

ie. graph has opposite sides with each cycle of π.

$\Rightarrow$ graph will have odd harmonics

Note: A graph being even or odd does affect nature of harmonics.

Find the fourier trigonometric series for f(t) (from previous graph) using half-range

$$f(t) = \begin{cases} -t & \text{if } -\pi \leq t < 0 \\ t & \text{if } 0 \leq t < \pi \end{cases}$$

$$\int t \cos nt\,dt = \frac{1}{n^2}(\cos nt + nt \sin nt)$$

Since function is even, $b_n = 0$

$L = \pi \longrightarrow L = \text{Period}/2$

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt$$

$$= \frac{2}{\pi}\int_0^\pi t\,dt$$

$$= \frac{2}{\pi}\left[\frac{t^2}{2}\right|_0^\pi$$

$$= \frac{2}{\pi}\frac{\pi^2}{2}$$

$$= \pi$$

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,dt$$

$$= \frac{2}{\pi}\int_0^\pi t\cos nt\,dt$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}(\cos nt + nt\sin nt)\right|_0^\pi$$

$$= \frac{2}{\pi n^2}\left[(\cos n\pi + 0) - (\overset{\cos 0}{\widehat{1}} + 0)\right]$$

$$= \frac{2}{\pi n^2}\left[(\cos n\pi - 1)\right]$$

both $= 1$ when n is even.

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right] \quad -①$$

If n is odd, $① = \frac{-4}{\pi n^2}$

I n is even, $= 0$

For the series, odd values of n are required to be generated.

$\Rightarrow$ Use $(2n-1) \Leftarrow n$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L}$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} -\frac{4}{\pi(2n-1)^2}\frac{\cos(2n-1)\cancel{\pi}t}{\cancel{\pi}}$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)t}{(2n-1)^2}$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{1}{9}\cos 3t + \frac{1}{25}\cos 5t + \dots\right)$$

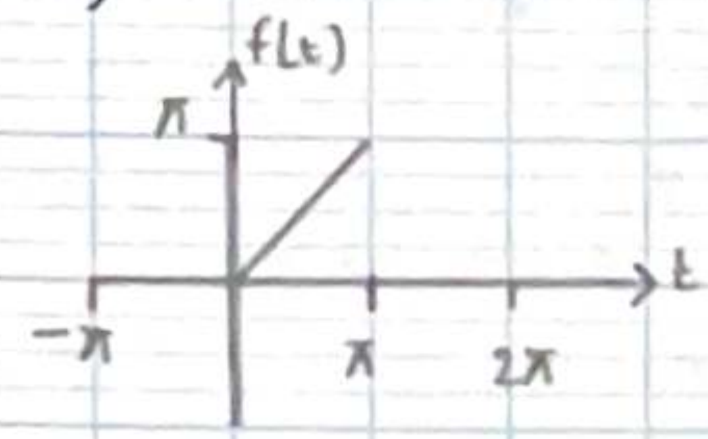
Adding more term will approx triangular function.

Odd functions and half-range Sine series.

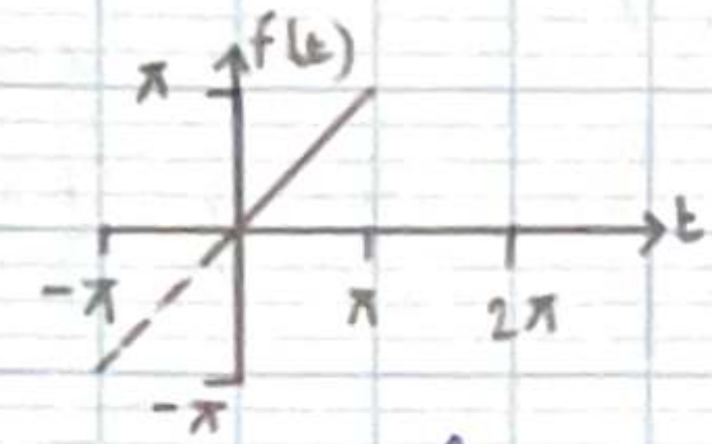
Since function is odd, a_0 & $a_n = 0$

$$b_n = \frac{2}{L}\int_0^L f(t) \frac{\sin n\pi t}{L} dt$$

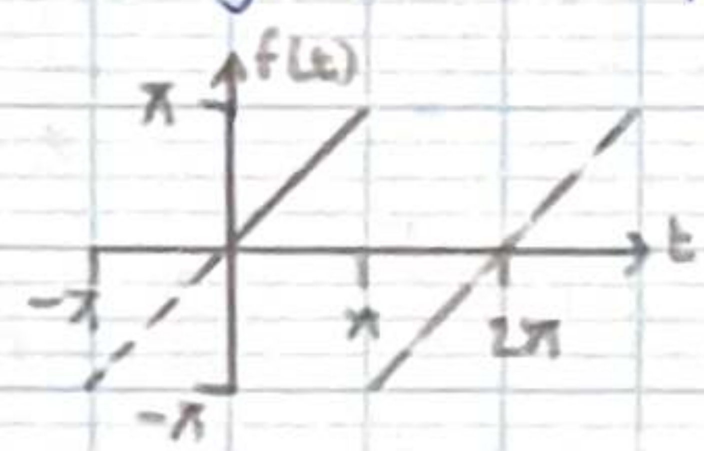
$f(t) = t$ from $t=0$ to $t=\pi$.



for function to be odd, it must be symmetrical about the origin.



Assuming waveform is periodic with period 2π,



$= \frac{2}{n^2}\left[\cancel{(\sin n\pi - n\pi \cos n\pi)} - (\right.$

period $= 2\pi \Rightarrow L = \pi$

$$b_n = \frac{2}{\pi}\int_0^\pi t \sin nt \, dt$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}\left(\sin nt - nt\cos nt\right|_0^\pi\right] \quad *$$

$$= \frac{2}{n^2}\left[\left(\underbrace{\sin n\pi}_{0} - n\pi\cos n\pi\right) - \left(\underbrace{\sin 0}_{0} - 0\right)\right]$$

$\rightarrow \sin\pi = 0$

$\sin 2\pi = 0$

$\vdots$

$\sin n\pi = 0$

$$= \frac{2}{n^2}$$

??

$$= \frac{-2}{n^{\cancel{3}}}\left(\cancel{n}\pi\cos n\pi\right)$$

$$= \frac{-2}{n}\pi\cos n\pi$$

Line Spectrum

The Sum of Sines and Cosines with the same period can be expressed as follows:

$a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)$ where $R = \sqrt{a^2 + b^2}$

$\alpha = \arctan\left(\frac{b}{a}\right)$

The fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

Can be written in harmonic form.

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega t - \varphi_n)$$

where $C_1 \cos(\omega t - \varphi_1)$ is the findamental

$C_2 \cos(2\omega t - \varphi_2)$ is the second harmonic

$\vdots$

φ_n = phase angle

C_n = harmonic amplitude = $\sqrt{(a_n)^2 + (b_n)^2}$

C_0 = Mean Value.

A plot showing the harmonic amplitude in the wave is called a line spectrum.

Plot the first 4 terms of the line spectrum for the fourier Series:

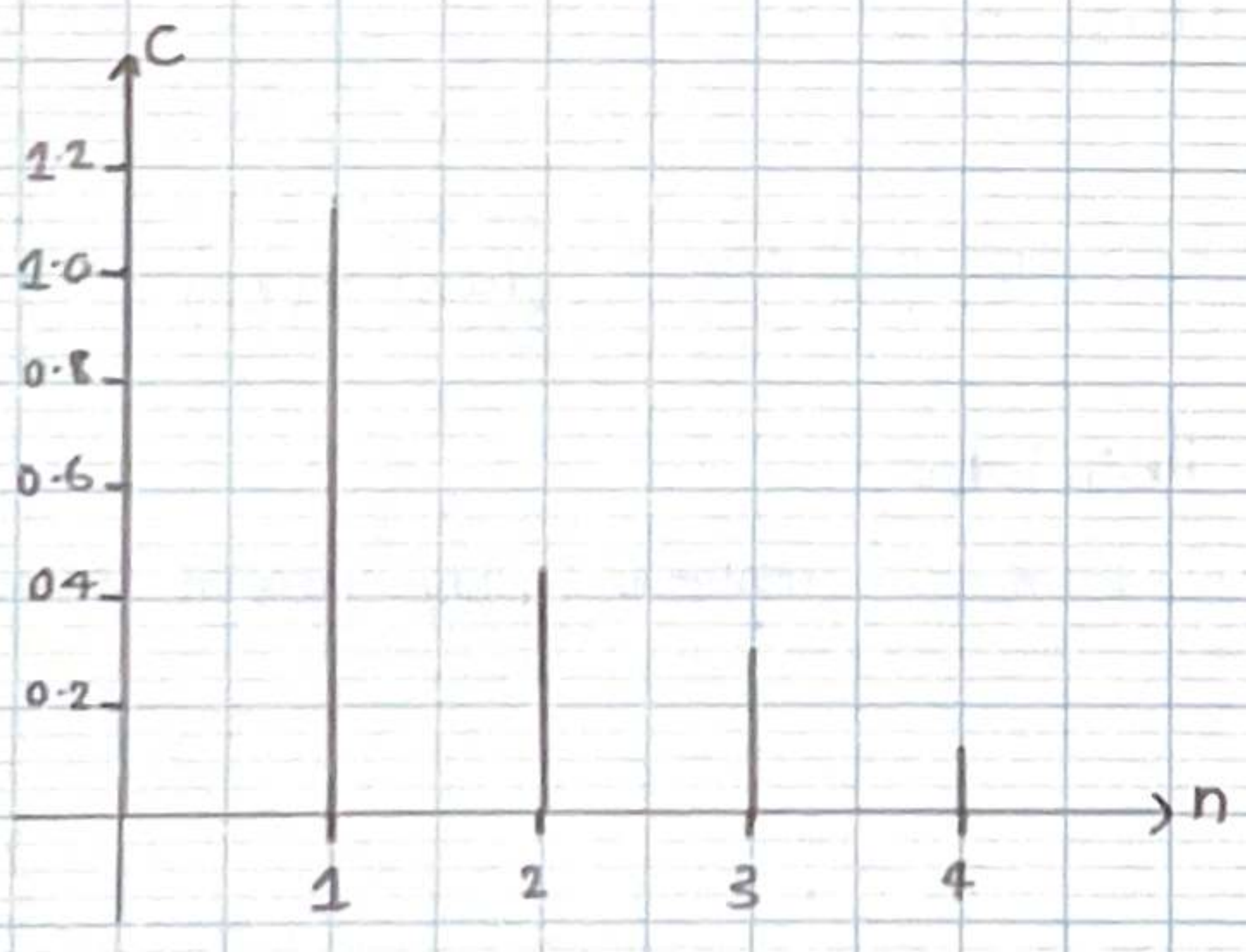
$$f(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{1}{2n-1} \cos nt + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \sin nt$$

$$a_n = \frac{1}{2n-1} \qquad b_n = \frac{(-1)^n}{2n}$$

$$C_n = \sqrt{(a_n)^2 + (b_n)^2}$$

n	$a_n = \frac{1}{2n-1}$	$b_n = \frac{(-1)^n}{2n}$	$C_n = \sqrt{a_n^2 + b_n^2}$
1	1	$-1/2$	1.118
2	$1/3$	$1/4$	0.4167
3	$1/5$	$-1/6$	0.260
4	$1/7$	$1/8$	0.190

Spectrum:



Line spectrum for f(t)

Fast Fourier Transform (FFT)

For a CD having a sampling rate of 44100/second, if the length of the recording is 1024 Samples,

⇒ amount of time represented by recording is $\frac{1024}{44100} = 0.02322\ S$

⇒ base frequency $f_0 = \frac{44100}{1024} = 43.066\ Hz$.

$\frac{1}{T}$

$2 \times 43.066 = 86.132\ Hz$ f_1

$3 \times 43.066 = 129.20 Hz$ f_2

⋮

Given $a_0 = 1.6$ $a_n = \frac{(-1)^n}{n}$, $b_n = \frac{1}{2n-1}$

For frequencies 43.066^{Hz}, 86.132^{Hz}, 129.20^{Hz}

we use $T = \frac{2\pi}{b}$

$b = \frac{44100(2\pi)}{1024} = 270.59$

fourier series would be ; $f(t) = \frac{1.6}{2} + \sum_{n=1}^{5} \left(\frac{(-1)^n}{n} \cos 270.59nt + \frac{1}{2n-1} \sin 270.59nt \right)$

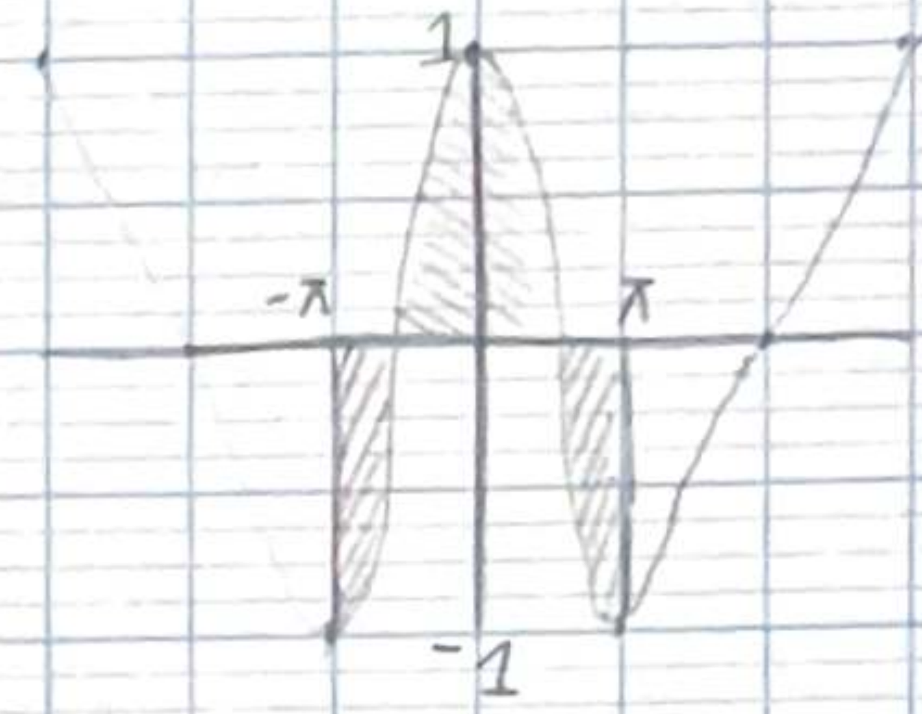
Expand.

Sound wave from digital data from CD was reconstructed

Revisions for fourier Series.

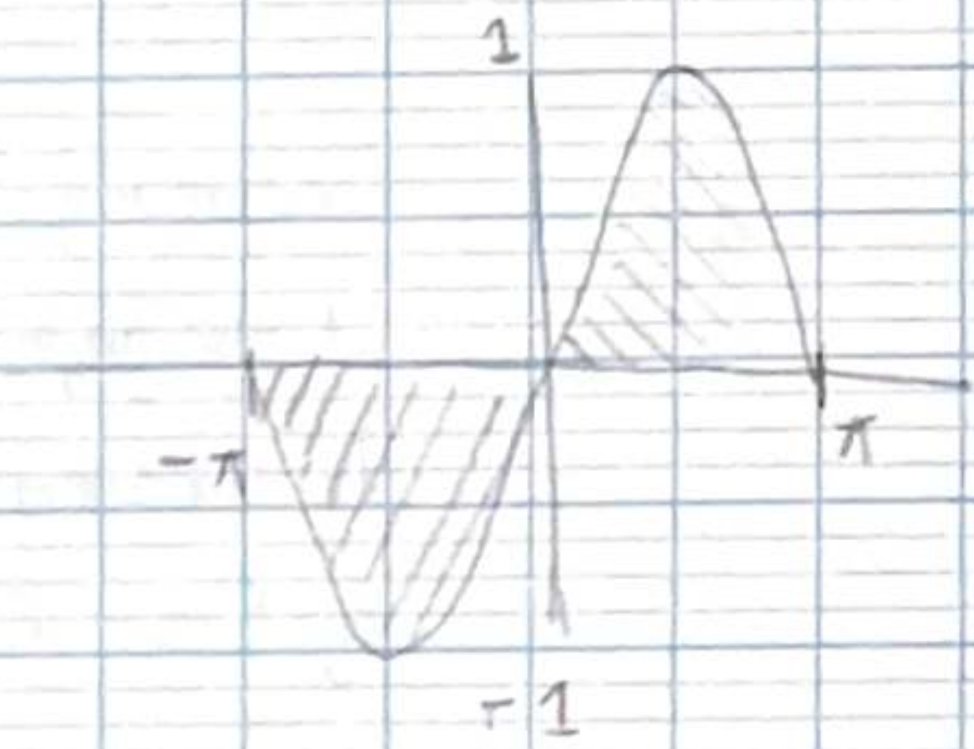
$f(x) = \cos(x)$ is said to be an even function as it is symmetrical about the vertical axis.

$\cos(-x) = \cos(x) \Rightarrow \boxed{\int_{-\pi}^{\pi} \cos\theta \, d\theta = 0}$



$f(x) = \sin(x)$ is an odd function. ie. it is symmetric about the origin.

$\sin(-x) = -\sin(x) \Rightarrow \boxed{\int_{-\pi}^{\pi} \sin\theta \, d\theta = 0}$



Multiples of π for Sine & Cosine Curves.

$\sin(n\pi) = 0$

$\sin\frac{(2n-1)}{2}\pi = (-1)^{n+1}$

$\cos(2n\pi) = 1$

$\cos[(2n-1)\pi] = -1$

$\cos(n\pi) = (-1)^n$

for $n = 0, 1, 2, 3$, $n \in \mathbb{Z}$

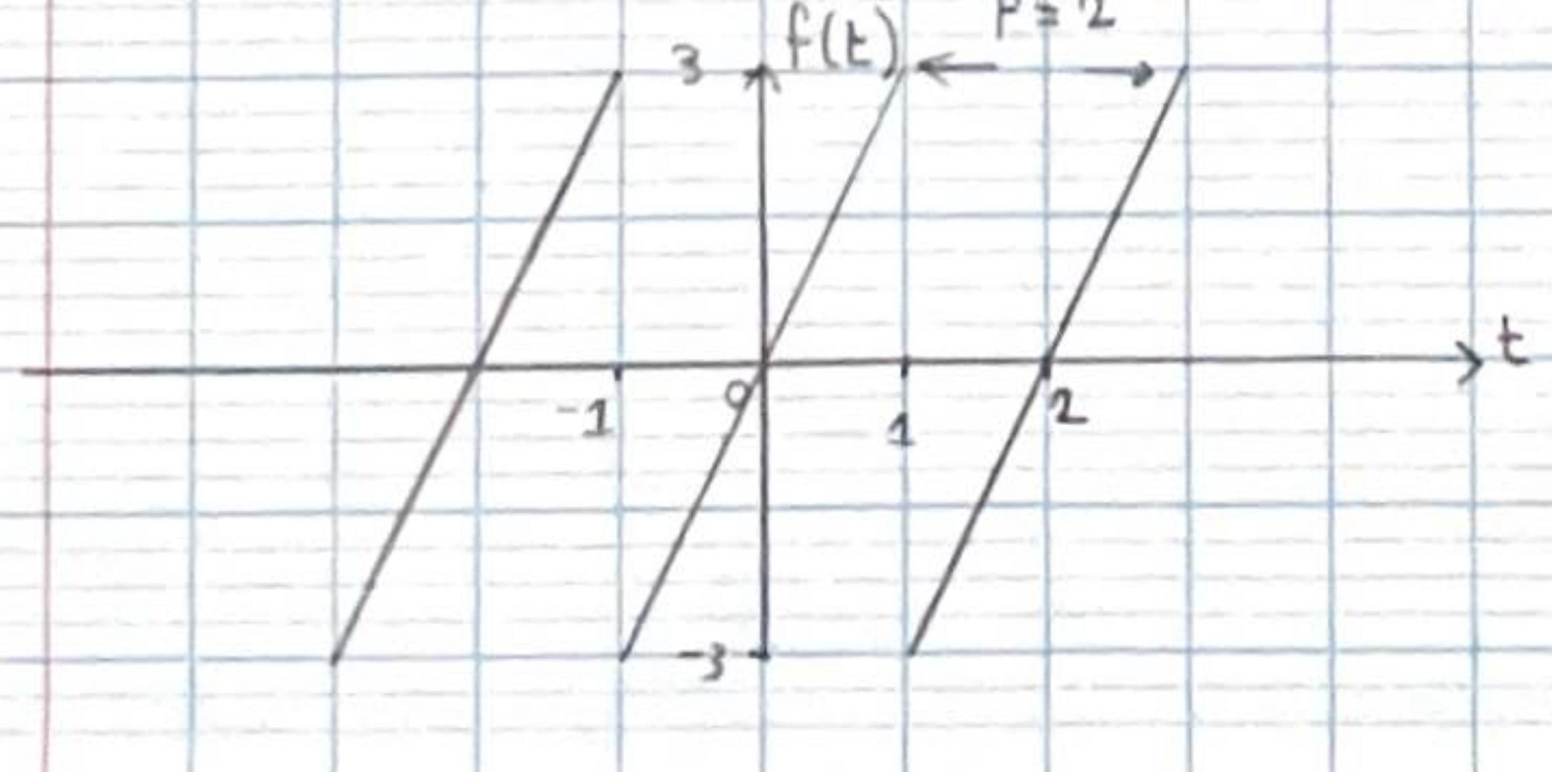
Periodic functions

A function $f(t)$ is said to be periodic with period p if

$f(t+p) = f(t) \quad \forall (t \wedge p) > 0$

↳ $p = 2\pi$ for $f(t) = \sin(x)$

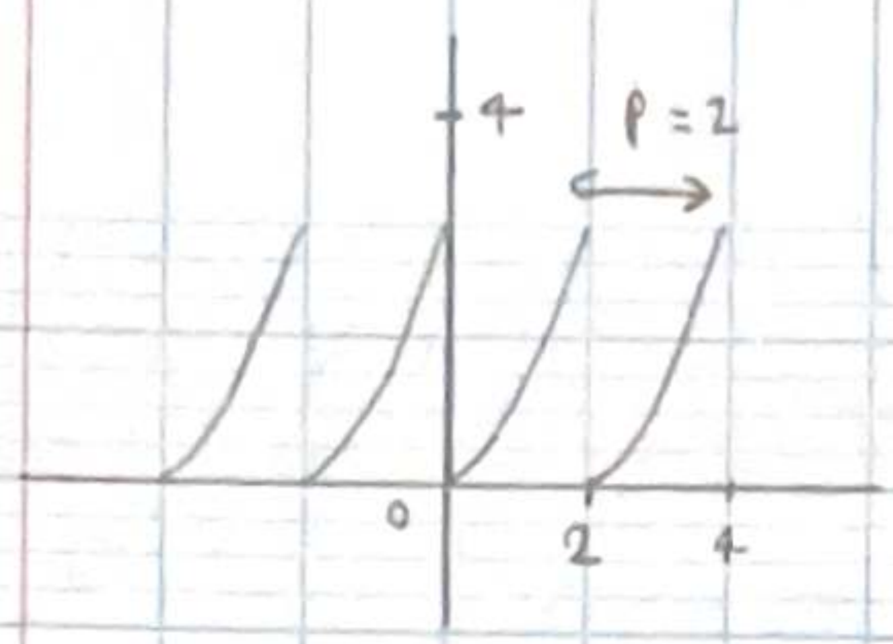
Sawtooth waveform with period $= 2$



$f(t) = 3t$ for $(-1 \leq t < 1)$

$f(t) = f(t+2)$

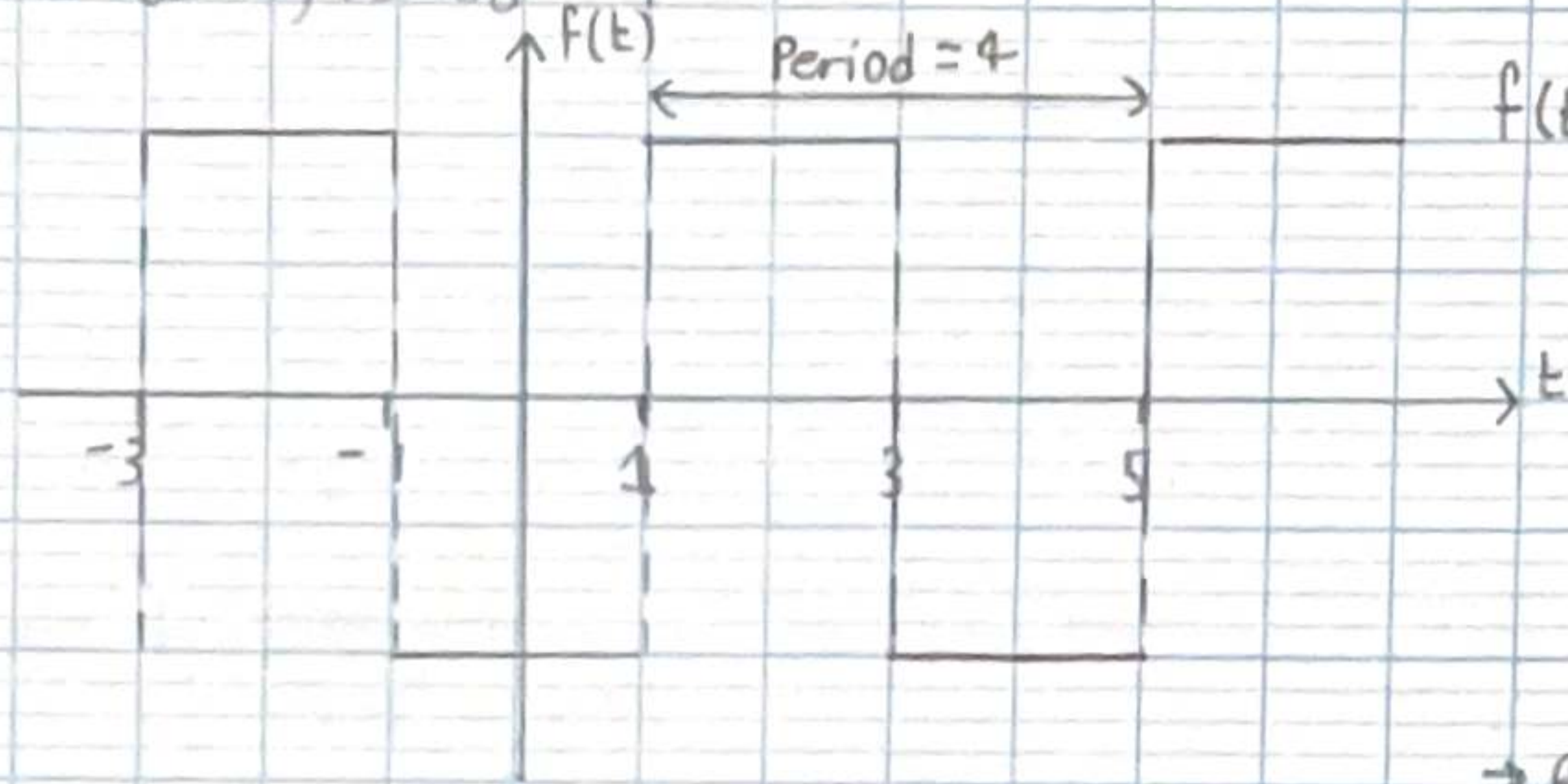
↳ period $= 2$



$f(t) = t^2$ (for $0 \le t < 2$)

$f(t) = f(t+2)$ [$\Rightarrow$ period $= 2$]

Square Wave, Period = 4



$f(t) = -3$ for $-1 \le t < 1$

and $1 \le t < 3$

$f(t) = f(t+4)$

$\Rightarrow$ Period = 4

Can be written as $2L = 4$

→ assume one cycle start at -2 & ends at 2 and the cycle goes from $-L$ to L

Continuity

If the graph of a function has no sudden jumps or breaks It is called a continuous function.

eg. Sine, Cosine, exponential, parabolic.

Finite discontinuity - A function that makes a finite jump at some point or points in an interval.

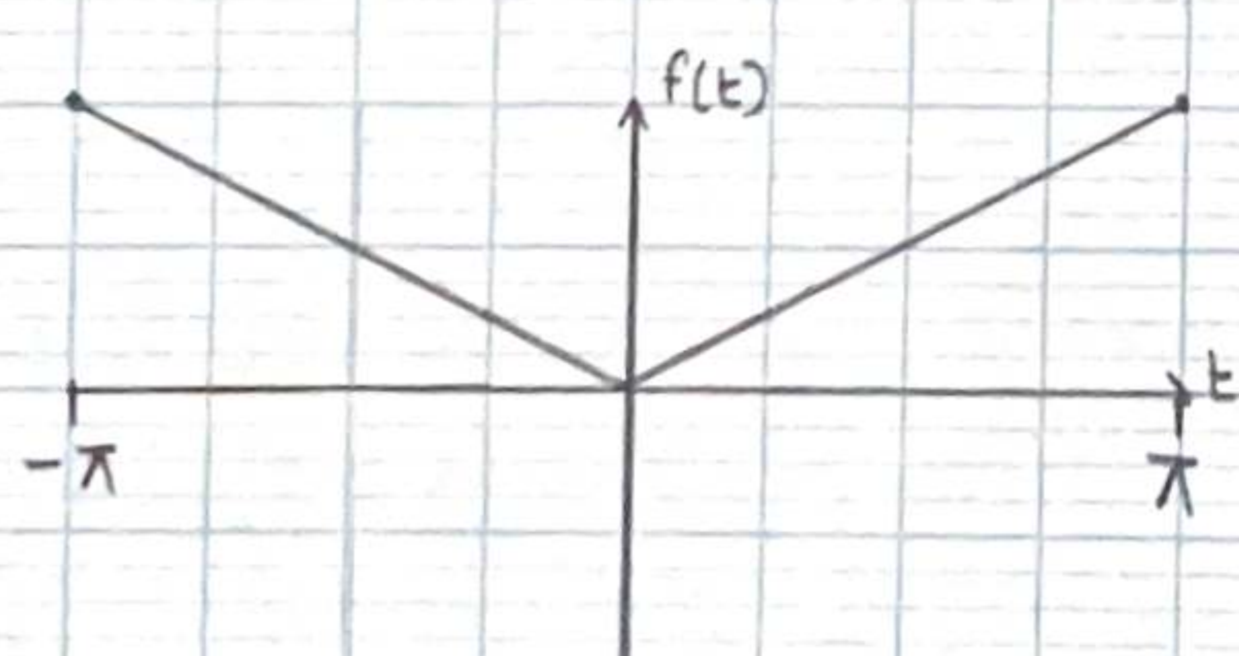
eg. Sawtooth, Square wave

Split functions

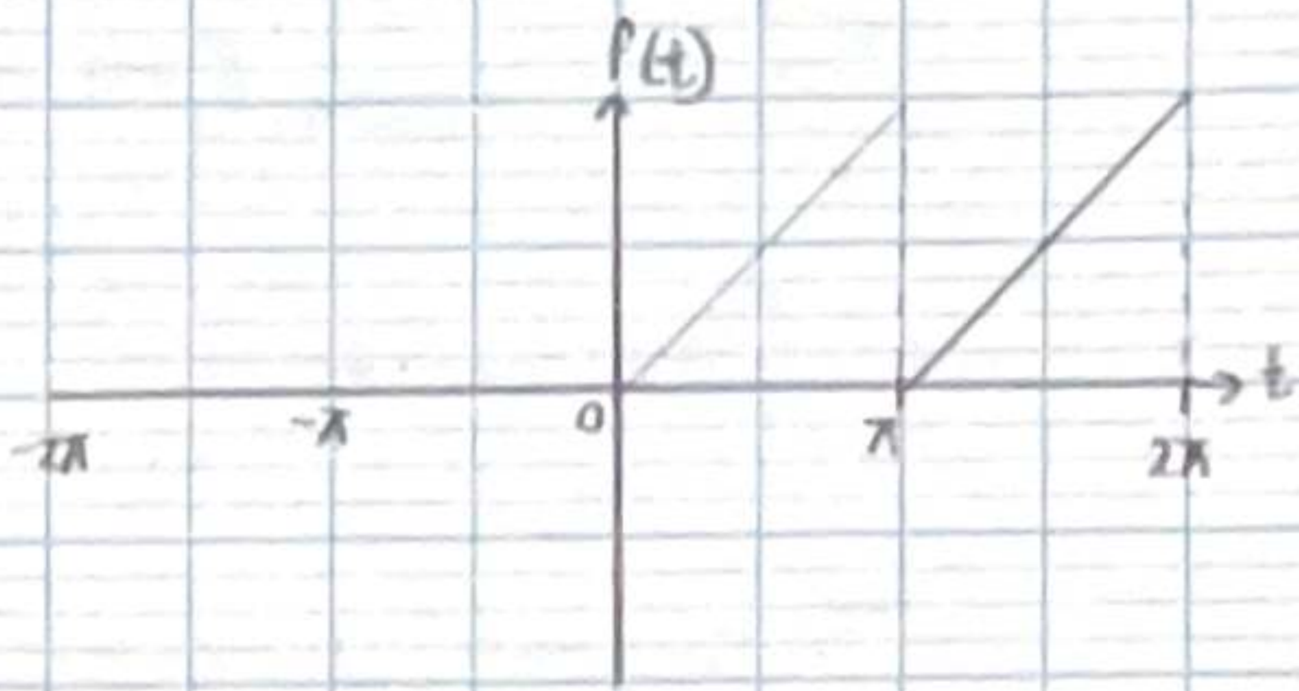
↳ Behaviour of charge, current, voltage in an AC can be described using this.

examples

for $f(t) = \begin{cases} -t & \text{if } -\pi \le t < 0 \\ t & \text{if } 0 \le t < \pi \end{cases}$



$$f(t) = \begin{cases} t & \text{if } 0 \leq t < \pi \\ t - \pi & \text{if } \pi \leq t < 2\pi \end{cases}$$



t.p at $-\pi$

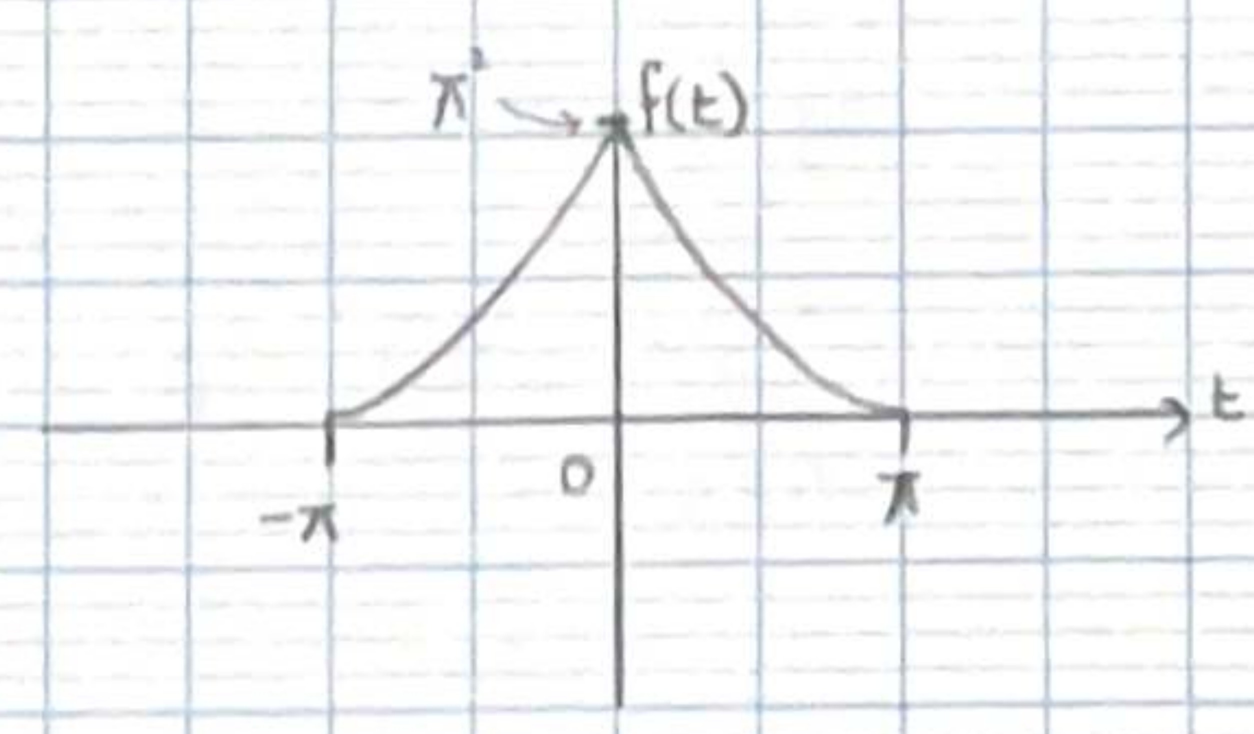
Period $= \pi$

$$f(t) = \begin{cases} (t+\pi)^2 & \text{if } -\pi < t < 0 \\ (t-\pi)^2 & \text{if } 0 \leq t < \pi \end{cases}$$

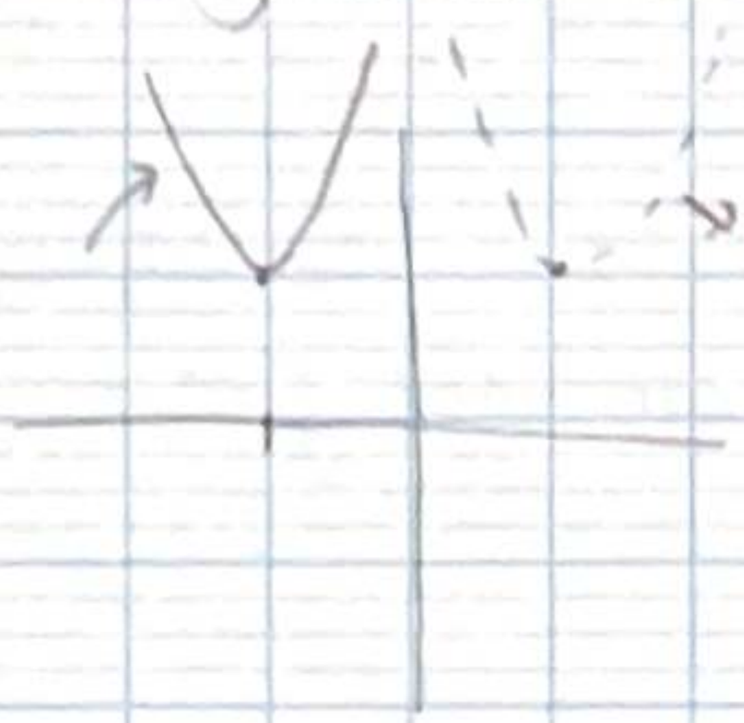
t.p at π

(min) parabolic equations

$t^2 \pm 2\pi t + \pi^2$

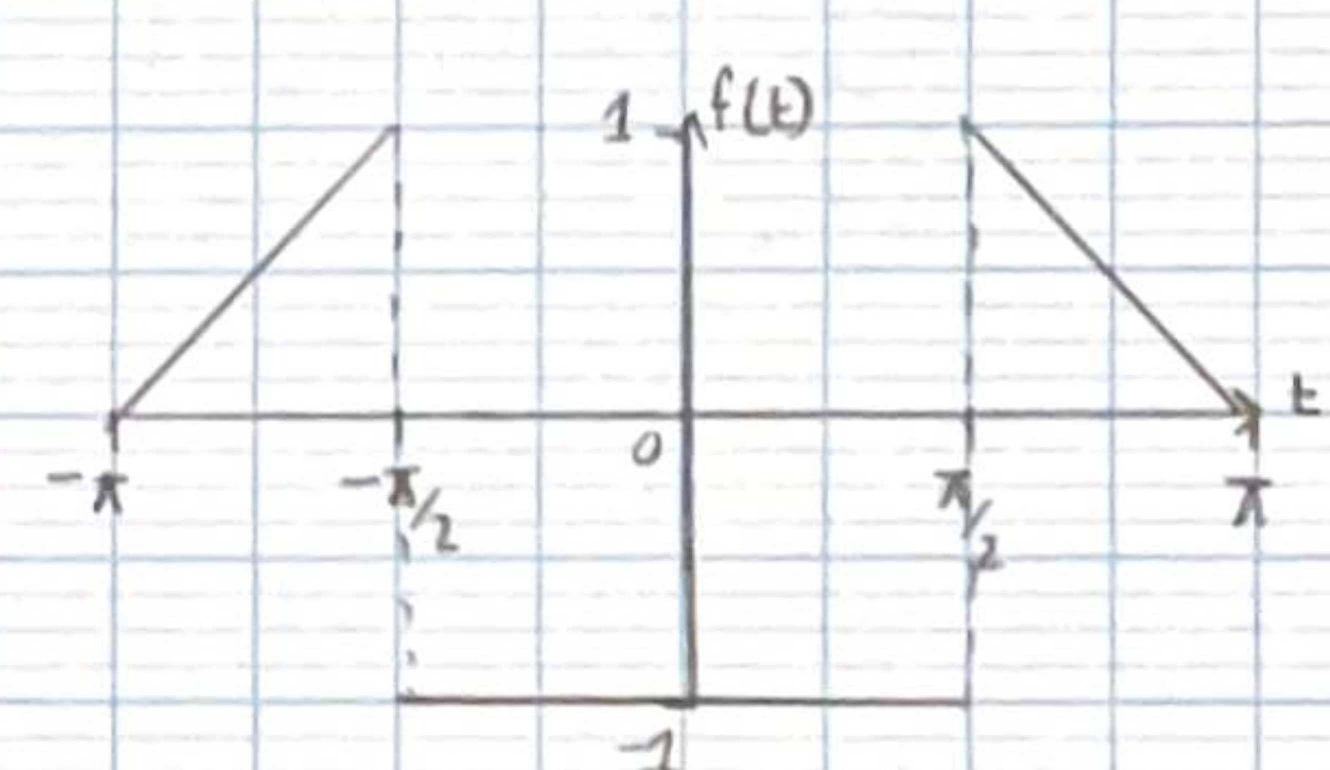


$t^2 + 2\pi t + \pi^2$



$t^2 - 2\pi t + \pi^2$

$$f(t) = \begin{cases} t + \pi & \text{if } -\pi \leq t < -\pi/2 \\ -1 & \text{if } -\pi/2 \leq t < \pi/2 \\ -t + \pi & \text{if } \pi/2 \leq t < \pi \end{cases}$$



Summation Notation

(i) $$\sum_{n=1}^{3} \frac{n}{n+1} = \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1}$$

$$= \frac{1}{2} + \frac{2}{3} + \frac{3}{4}$$

$$= \frac{6}{12} + \frac{8}{12} + \frac{9}{12} = \frac{23}{12}$$

(ii) $$\sum_{n=1}^{5} (2n-1) = 1 + 3 + 5 + 7 + 9$$

$$= 25$$

This generates odd integers

$$\sum_{n=1}^{5} 2n \longrightarrow 2+4+6+8+10=30$$

Generates even integers.

To generate alternate positive & negative numbers, we multiply the expression in the summation by $(-1)^{n+1}$

eg.

$$\sum_{n=1}^{5} (-1)^{n+1}(n) = [(1)(1)] + [(-1)(2)] + \dots$$

$$= 1-2+3-4+5$$

$$= 3$$

example. ~~Notation~~ Expression giving alternate positive & negative fractions with even denominators.

$$\sum_{n=1}^{5} (-1)^{n+1} \cdot \frac{1}{2n} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10}$$

$$= 47/120$$

(iv) $$\sum_{n=1}^{5} n^2 a_n = a_1 + 4a_2 + 9a_3 + 16a_4 + 25a_5$$

(v) $$\sum_{n=1}^{4} \frac{n\pi t}{L} = \frac{\pi t}{L} + \frac{2\pi t}{L} + \frac{3\pi t}{L} + \frac{4\pi t}{L}$$

$$= \frac{10\pi t}{L}$$

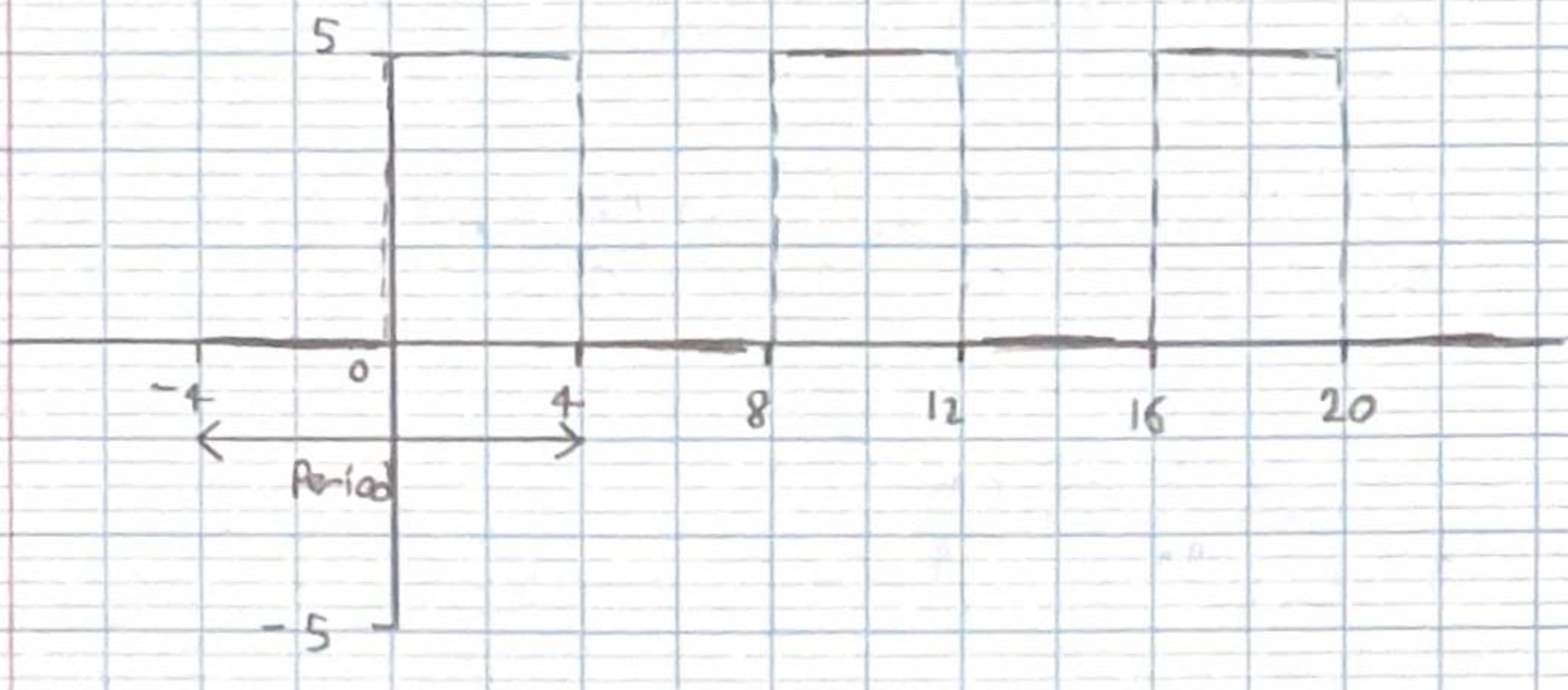
or $\Rightarrow \frac{\pi t}{L}(n) = \frac{\pi t}{L}\sum_{n=1}^{4}(n) = 10\frac{\pi t}{L}$

$$f(t) = \begin{cases} 0 & \text{if } -4 \le t < 0 \\ 5 & \text{if } 0 \le t < 4 \end{cases} \qquad f(t) = f(t+8)$$

Period $= 8$

rectangular function

Sketch for 3 cycles



Evaluating fourier Coefficients.

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt$$

$2L = p = 8$

$L = 4$

$$a_0 = \frac{1}{4}\int_{-4}^{4} f(t)\,dt$$

$$= \frac{1}{4}\left[\int_{-4}^{0}(0)\,dt + \int_{0}^{4}(5)\right]$$

$$= \frac{1}{4}\left[0 + \left[5t\right|_0^4\right]$$

$$= \frac{1}{4}(20)$$

$$= 5 \leftarrow$$

$$\text{Mean Value} = \frac{a_0}{2} = 5/2$$

Can determine a_0 from graph.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt$$

$$= \frac{1}{4}\int_{-4}^{4} f(t)\cos\frac{n\pi t}{4}\,dt$$

$$= \frac{1}{4}\left[\int_{-4}^{0} (0)\cos\frac{n\pi t}{4}\,dt + \int_{0}^{4} (5)\,\frac{n\pi t}{4}\,dt\right]$$

$$= \frac{1}{4}\left[0 + \left[5\,\frac{4}{n\pi}\sin\frac{n\pi t}{4}\right|_{0}^{4}\right]$$

$$= \frac{5}{n\pi}\left(\sin\frac{n\pi 4}{4} - \sin\frac{n\pi(0)}{4}\right)$$

$$= \frac{5}{n\pi}(\sin n\pi - 0)$$

$$= 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt$$

$$= \frac{1}{4}\int_{-4}^{4} f(t)\sin\frac{n\pi t}{4}\,dt$$

$$= \frac{1}{4}\left(0 + \left[(-5)\left(\frac{4}{n\pi}\right)\left[\cos\frac{n\pi t}{4}\right|_{0}^{4}\right]\right.$$

$$= \frac{-5}{n\pi}\left(\cos\frac{n\pi 4}{4} - 1\right)$$

$$= -\frac{5}{n\pi}(\cos n\pi - 1)$$

Substituting into fourier series,

$$f(t) = \frac{a_0}{2} \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

$$= \frac{5}{2} + \sum_{n=1}^{\infty} (0)\cos\frac{n\pi t}{4} + \sum_{n=1}^{\infty} -\frac{5}{n\pi}(\cos n\pi - 1)\sin\frac{n\pi t}{4}$$

$$= 2.5 - \frac{5}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}$$

Substitute $n = 1, 2, 3, \ldots$ into series

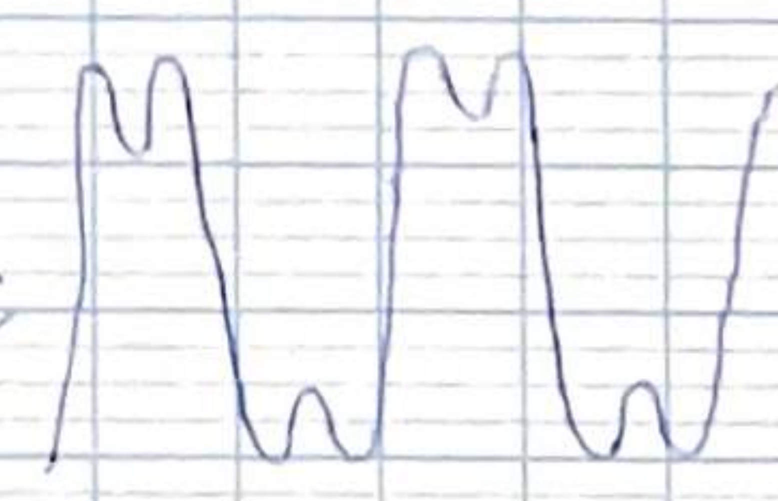
$$f(t) = 2.5 + \frac{10}{\pi}\left(\underbrace{\sin\frac{\pi t}{4}}_{1} + \underbrace{\frac{1}{3}\sin\frac{3\pi t}{4}}_{2} + \underbrace{\frac{1}{5}\sin\frac{5\pi t}{4}}_{3} + \cdots\right)$$

n	$\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}$
1	$\frac{1}{1}(\cos\pi - 1)\sin\frac{\pi t}{4} = -2\sin\frac{\pi t}{4}$
2	$\frac{1}{2}(\cos 2\pi - 1)\sin\frac{\pi t}{2} = 0$
3	$\frac{1}{3}(\cos 3\pi - 1)\sin\frac{3\pi t}{4} = -\frac{2}{3}\sin\frac{3\pi t}{4}$
⋮	⋮

2.5 +

1 $\frac{10}{\pi}\sin\frac{\pi t}{4} \rightarrow$

(+)



original graph +

2 $\frac{10}{3\pi}\sin\left(\frac{3\pi t}{4}\right) \rightarrow$

More terms implies closer to 'real' graph.