Circle Group

A group whose operation is identified by multiplication

The circle group ($\mathbb{T}$) is the multiplicative group of all complex numbers with absolute value 1 (ie. the unit circle in the complex plane)

$$\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$$

Let $n \in \mathbb{Z}$ be an integer such that $n > 0$

Let $U_n = \{ z \in \mathbb{C} : z^n = 1 \}$ denote the set of complex n^{th} roots of unity

Let $(U_n, \cdot)$ be the algebraic structure formed by U_n under complex multiplication.

Then $(U_n, \cdot)$ is the multiplicative group of complex n^{th} roots of unity

The circle group forms a subgroup of $\mathbb{C}^{\times}$, the multiplicative group of all non-zero complex numbers.

Roots of unity under multiplication form Cyclic group.

$$U_n = \{ e^{2ik\pi/n} \; k \in \mathbb{N}_n \}$$

↑ set of complex n^{th} roots of unity.

Let $\omega = e^{2i\pi/n}$

Then we have $U_n = \{ \omega^k : k \in \mathbb{N}_n \}$

that is

$$U_n = \{ \omega^0, \omega^1, \omega^2, \ldots \omega^{n-1} \} \rightarrow \omega^n = e^{2\pi i} = 1 = \omega^0$$

Let $\omega^a, \omega^b \in U_n$

Then $\omega^a \omega^b = \omega^{a+b} \in U_n$

Either $a+b < n$ in which case $\omega^{a+b} \in U_n$ or $a+b \geq n$ in which case

$$\begin{aligned} \omega^a \omega^b &= \omega^{a+b} \\ &= \omega^{n+t} \quad \text{for some } t < n \\ &= \omega^n \omega^t \\ &= \omega^t \quad \text{as } \omega^n = 1 \end{aligned}$$

So U_n is closed under multiplication

$\omega^0 = 1$ which is the identity

ω^{n-t} is the Inverse of ω^t

Finally we note than U_n is generated by ω $\Rightarrow$ Cyclic grop

Hence by definition of cyclic group and cyclic groups of the same order are isomorphic

$U_n = \langle \omega \rangle \cong C_n$

Cardinality

Two sets (either finite or infinite) which are equivalent are said ~~tho~~ to have the same cardinality.

The cardinality of set S is written as $|S|$

The order of an algebraic structure $(S, \circ)$ is the cardinality $|S|$ of its underlying set S.

Cardinality of a finite set.

Let S be a finite set

The cardinality $|S|$ of S is the no of elements in S.

That is if:

$S \sim \mathbb{N}_{<n}$ where $\sim$ denotes set equivalence

$\mathbb{N}_{<n}$ denotes the set of natural numbers $< n$

$\underbrace{\text{Iso}}_{\text{greek for equal}}\underbrace{\text{morphism}}_{\text{greek for structure}}$

$\downarrow$

equal structure.

Cardinality of an infinite set.

Let S be an infinite set

$|S| = \infty$

Cannot be assigned a number $n \in \mathbb{N}$

$\Rightarrow |S|$ is at least $\aleph_0$ (aleph null)