4/ Is $(S_2, \circ)$ a Commutative group?

Note is $D_2 \cong S_2$
Can this be used?

$S_2 = \{1, 2\}$

$f = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad g = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = e$

$\Rightarrow f \circ e = e \circ f = f$

$\Rightarrow$ Commutativity is Satisfied (trivially)

5 (a) There are 4 elements in S_4.

(b) Calculate the composition of.

i) $\overset{a}{(1\ 2\ 4)(3)} \circ \overset{b}{(1\ 2)(3)(4)}$

$a \circ b = (1\ 4)\cancel{(2)}\cancel{(3)}$

(ii) $b \circ d$

$(1\ 2) \circ (1\ 2\ 4)$

1 remains unchanged ⇒ Start with 2

$= (1)(2\ 4)$

$= (2\ 4)$

Is S_4 a Commutative group?

$a \circ (b \circ d) \overset{!}{=} (a \circ b) \circ d$

$(1\ 4) \circ \cancel{(2\ 4)}$

$(1\ 2\ 4) \circ (2\ 4) \overset{!}{=} (1\ 4) \circ (1\ 2\ 4)$

$\downarrow \qquad \downarrow$

$(1\ 2) = (1\ 2)$

$\Rightarrow (S_4, \circ)$ is a Commutative groups

Are all S_n where $n = 2m$ where $m \in \mathbb{N}$ (even no. of symbols) Commutative?

(d) $f = (1\ 3\ 2\ 4)$

$f^{-1} = (4\ 2\ 3\ 1)$

$= (1\ 4\ 2\ 3) \leftarrow$

$g = (2\ 3\ 4)$

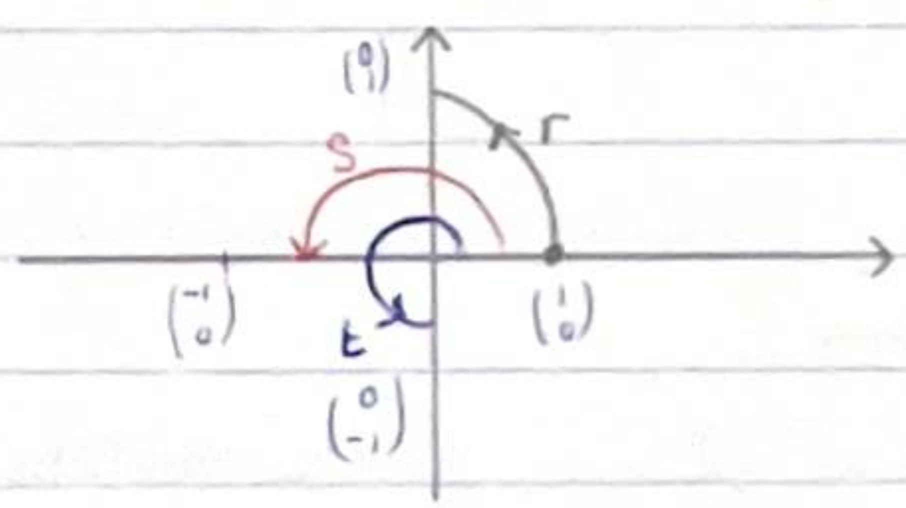
$g^{-1} = (4\ 3\ 2)$

$= (2\ 4\ 3) \leftarrow$

Isometry and Matrix groups.

- Rigid transformations of the plane (reflection, rotation, translations and their combinations) are called isometries
- Set of isometries form groups and when expressed using matrices give rise to matrix groups.

Consider the following isometries of the plane.



e = Identity / 360°

$r = 90^\circ$ cc/w rotation

$s = 180^\circ$ cc/w rotation

$t = 270^\circ$ cc/w rotation

$\circ$	e	r	s	t
e	e	r	s	t
r	r	s	t	e
s	s	t	e	r
t	t	e	r	s

From table we get: Closure

Identity

Invertibility

Associativity also holds

(Also, the set of isometries is ~~asso~~ commutative.

$r \circ s = t = s \circ r$

⇒ ~~Asso~~ Commutative.

??

⇒ $(e, r, s, t, \circ)$ is a group

A set of isometries which form a group under composition ~~are~~ is called an isometry group.

If we express each of the previous transformation in the form of matrices then the composition of transformations corresponds to matrix multiplication and form a group under matrix multiplication.

eg. Write down the matrix group corresponding to the previous group and verify that $r \circ s = t$ and $t^{-1} = r$

✓ $e \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓ $r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$: $\binom{1}{0} \mapsto \binom{0}{1}$ $\binom{0}{1} \mapsto \binom{-1}{0}$

✓ $s \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$: $\binom{1}{0} \mapsto \binom{-1}{0}$ $\binom{0}{1} \mapsto \binom{0}{-1}$

✓ $t \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: $\binom{1}{0} \mapsto \binom{0}{-1}$ $\binom{0}{1} \mapsto \binom{1}{0}$

⇒ the corresponding matrix group is $\left(\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \times \right)$

where $\times$ denotes matrix multiplication.

Do s first

$r \circ s$ corresponds to

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = t$$

Matrix Multiplication

$$t^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = r$$

Inverse of a Matrix.

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

change sign.

Interchange

Note: For a matrix to have an inverse, it must be non-singular ie. $\det(A) \neq 0$

$$A^{-1} = \frac{1}{\det(A)} \times \text{adj}.(A) = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Adjoint.

Note $\boxed{AA^{-1} = A^{-1}A = I}$

Satisfies invertibility property of Group.

$r \circ s = t$

$s \circ r = t$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = t$$

$\Rightarrow$ The matrix multiplication for this isometry group is commutative.

NOT all matrix groups arise from isometries as shown in the following example.

Show that the set of matrices in the form $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ where $n \in \mathbb{Z}$ forms a group under matrix multiplication. Interpret the group geometrically.

1) Closure

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+m \\ 0 & 1 \end{pmatrix} \quad (n+m) \in \mathbb{Z}$$

2) Identity

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \text{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3) Invertibility

$$\exists A \in S \mid A\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} = e$$

$$A = \begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix}1 & -n\\ 0 & 1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1 & -n\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

4) Associativity

$$\left[\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\right]\begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix} = \left[\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}\right]\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & k+m+n\\ 0 & 1\end{pmatrix}$$

$$\Rightarrow \left(\left\{\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} \mid n \in \mathbb{Z}\ \cancel{m\times}\right\}, \times\right) \text{ is a group.}$$

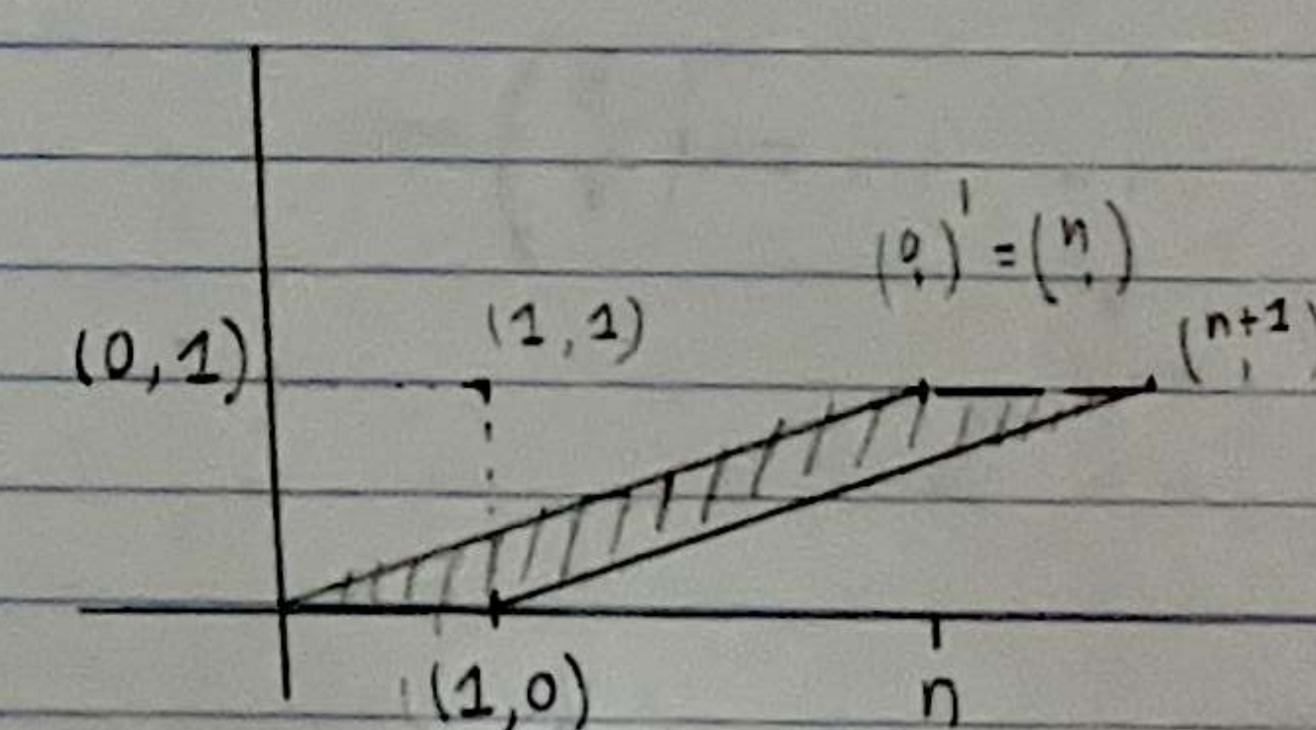
5) Commutativity

$$\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & m+n\\ 0 & 1\end{pmatrix}$$

$\Rightarrow$ Group is Abelian.

Geometric interpretation.

$$\Rightarrow \begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} : \begin{pmatrix}1\\0\end{pmatrix} \longmapsto \begin{pmatrix}1\\0\end{pmatrix},\ \begin{pmatrix}0\\1\end{pmatrix} \longmapsto \begin{pmatrix}n\\1\end{pmatrix}$$



Group of shears in the direction.

Consider transformation $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = T$

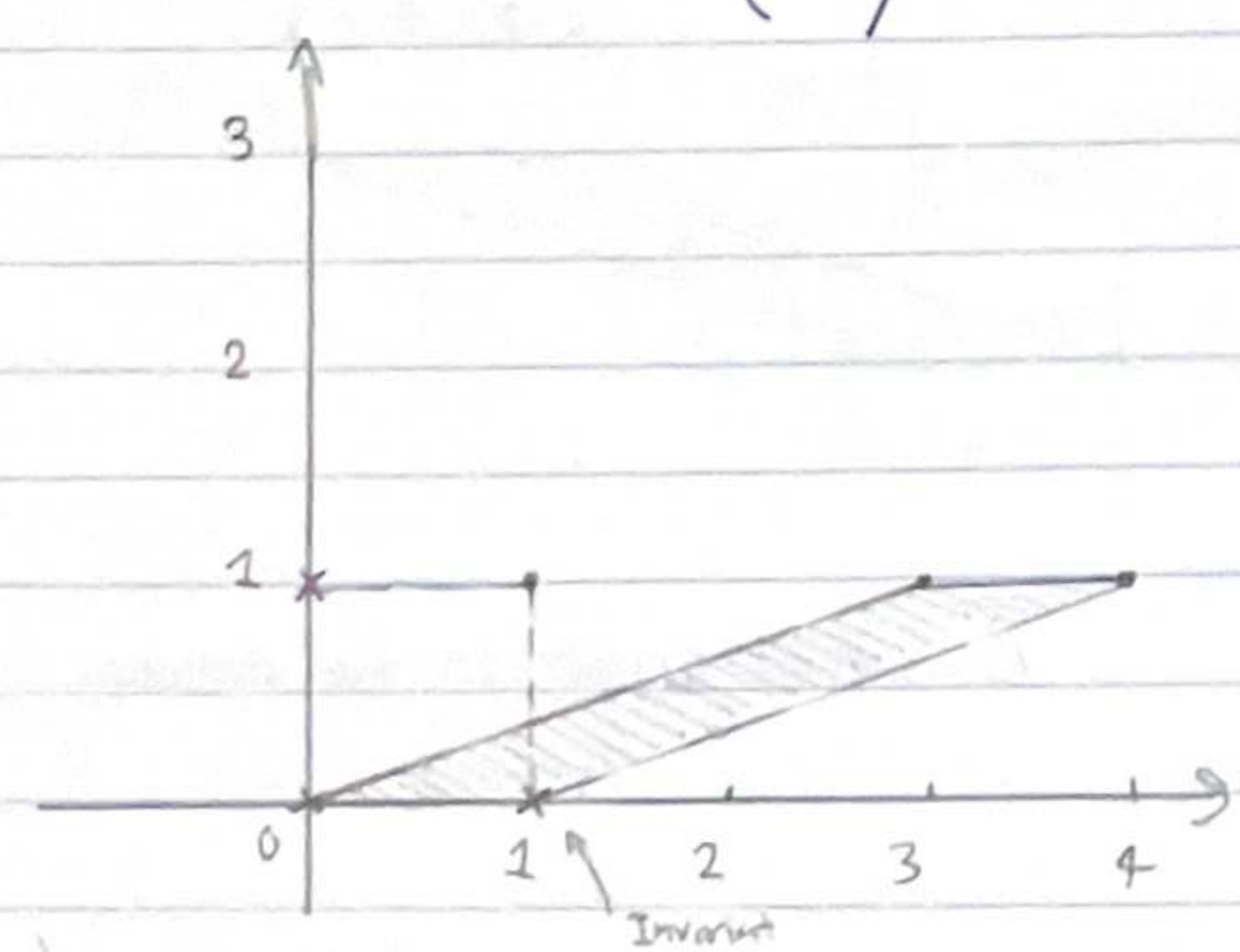
$\Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xmapsto{T} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xmapsto{T} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$?? ✓✓

ie. $(1\ \ 0)\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \quad (0\ \ 1)\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

$= (1\ \ 3) \quad = (0\ \ 1)$

✓✓ $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$



$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

1) Verify that the transformation

$$S = \left\{ \begin{array}{l} e = id \\ r = cc/w\ 120^\circ \\ s = cc/w\ 240^\circ \end{array} \right\} \text{(about origin)}$$

form a group.

Express the transformations in matrix form and verify $r \circ s = e$

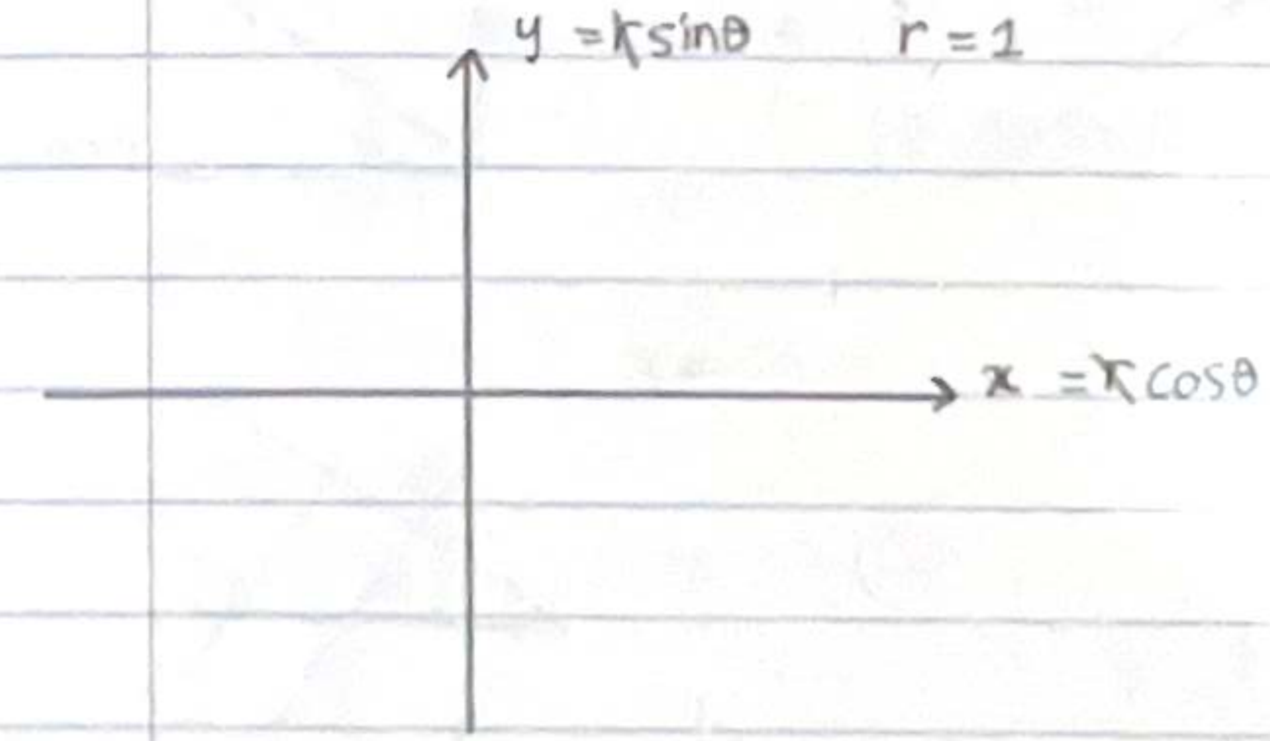
$r \circ s = e \in S$ ⇒ Closure.
⇒ Invertibility
⇒ Identity
⇒ Associative (trivially)
⇒ Abelian.

$\theta = 0^\circ = id \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \cancel{\begin{pmatrix} \cos(0) & 0 \\ 0 & \sin(0) \end{pmatrix}}$

$\equiv \cancel{\begin{pmatrix} \cos\theta & 0 \\ 0 & \cos(\theta+90)^\circ \end{pmatrix}}$

$= \begin{pmatrix} \cos 0 & 0 \\ 0 & \sin 90 \end{pmatrix}$

$= \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos(\theta+90) \end{pmatrix}$



$r = \begin{pmatrix} \cos 120 & 0 \\ 0 & \cos 210 \end{pmatrix} = \begin{pmatrix} -1/2 & 0 \\ 0 & -\sqrt{3}/2 \end{pmatrix}$ ✓

$s = \begin{pmatrix} \cos 240 & 0 \\ 0 & \cos 330 \end{pmatrix} = \begin{pmatrix} -1/2 & 0 \\ 0 & \sqrt{3}/2 \end{pmatrix}$ ✓

$\Rightarrow \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ Any point $\Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ \sin\theta \end{pmatrix}$

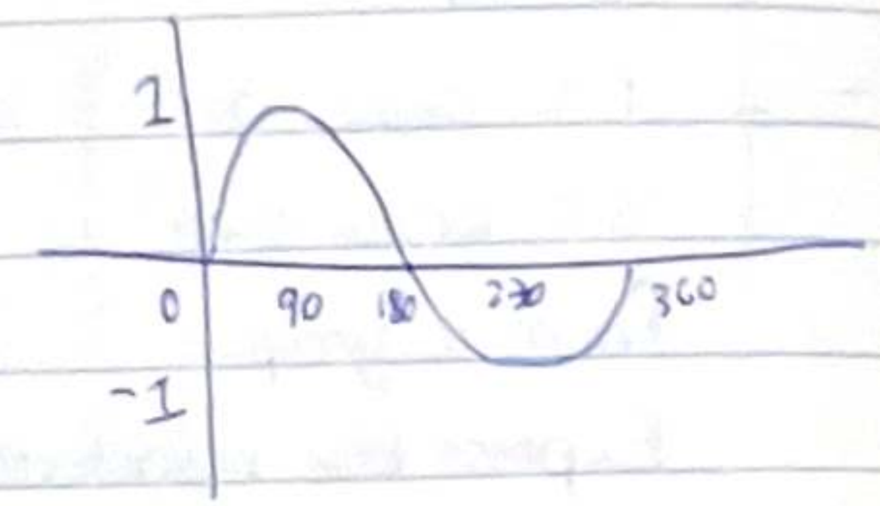
$T = \begin{pmatrix} \cos\theta & 0 \\ 0 & \sin\theta \end{pmatrix}$

$120 + 90 = 210$

$24 \ldots 33$

$$r \circ S = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cancel{\frac{1}{4}} & 0 \\ 0 & \cancel{\frac{3}{4}} \end{pmatrix}$$

$$= \cancel{\tfrac{1}{2}} \begin{pmatrix} -1 & 0 \\ 0 & -\sqrt{3} \end{pmatrix} \cancel{\tfrac{1}{2}} \begin{pmatrix} -1 & 0 \\ 0 & \sqrt{3} \end{pmatrix}$$



Verify that the transformation

$e = id$

$r = 120^\circ$ CC/w rotation

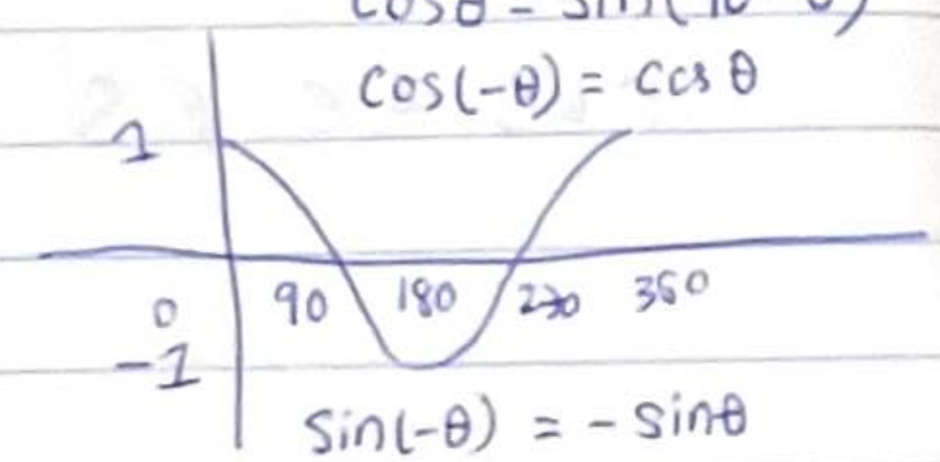
$S = 240^\circ$ CC/w rotation

forms a group.

	30	45	60
s	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
c	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
t	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$

$\cos\theta = \sin(90-\theta)$

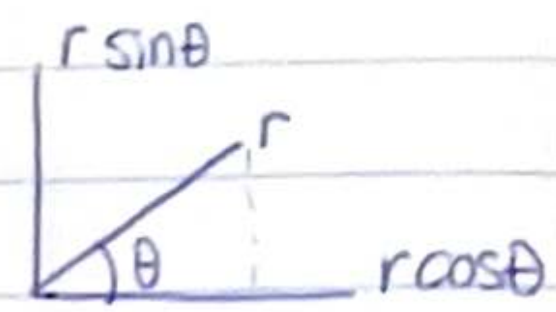
$\cos(-\theta) = \cos\theta$



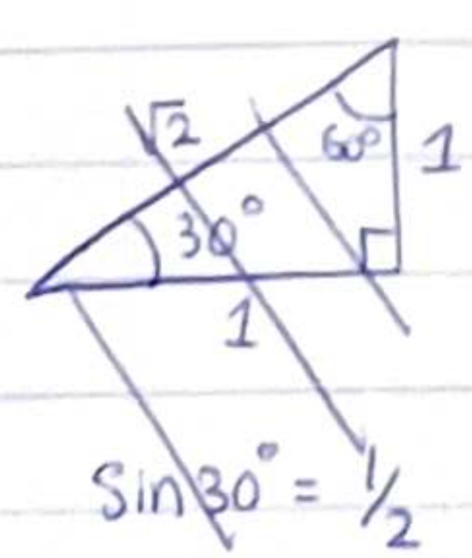
$\sin(-\theta) = -\sin\theta$

Express the transformation in matrix form and verify $r \circ S = e$

$$e = id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cancel{\cos\theta} & 0 \\ 0 & \cancel{\sin(90-\theta)} \end{pmatrix}$$



$$= \begin{pmatrix} \cos\theta & 0 \\ 0 & \sin(\theta+90) \end{pmatrix}$$



$\sin 30^\circ = \frac{1}{2}$

$$r = \begin{pmatrix} \cos 120 & 0 \\ 0 & \sin(210) \end{pmatrix}$$

$$\begin{array}{r} 1 \\ 240 \\ +\ 90 \\ \hline 120 \end{array}$$

$$= \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

$$S = \begin{pmatrix} \cos 240^\circ & 0 \\ 0 & \sin(330) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

$$r \circ S = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

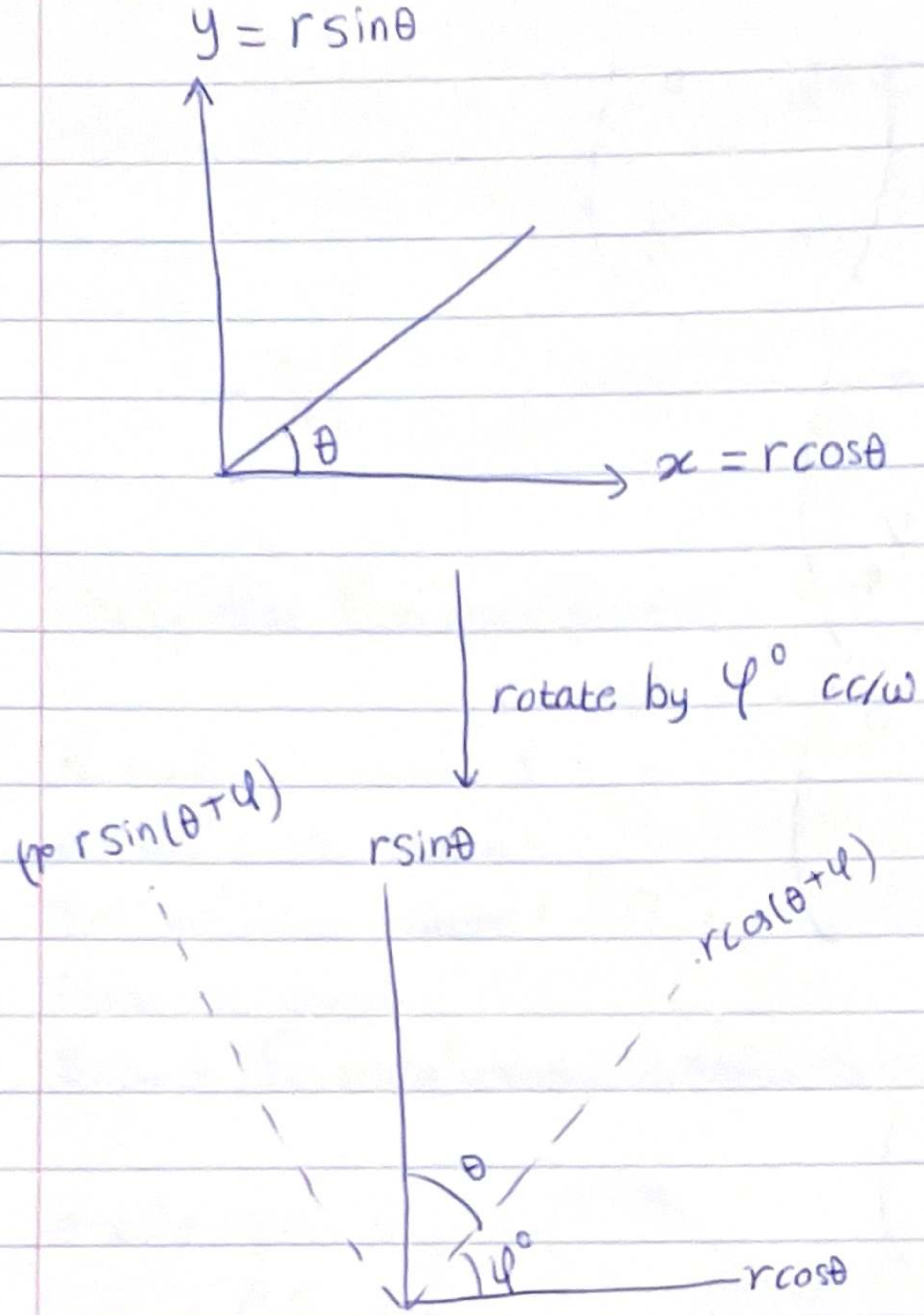
$$= \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$r = \begin{pmatrix} \cos 120 & -\sin 120 \\ \sin 120 & \cos 120 \end{pmatrix} \qquad S = \begin{pmatrix} \cos 240 & -\sin 240 \\ \sin 240 & \cos 240 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \qquad = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$


y = r sinθ
θ
x = r cosθ
rotate by φ° cc/w
r sin(θ+φ)
r sinθ
r cos(θ+φ)
θ
φ°
r cosθ

1/ Verify that the transformations

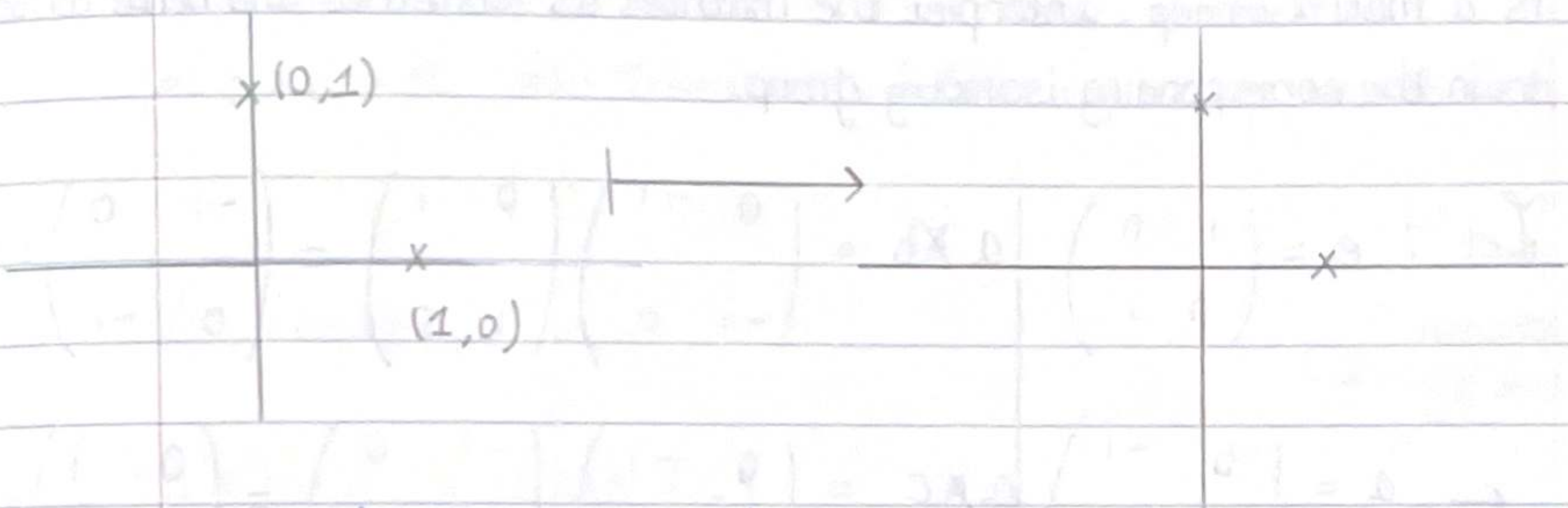
$$S = \begin{cases} e = id \\ r = cc/w\ 120^\circ \\ s = cc/w\ 240^\circ \end{cases}$$

form a group.

$S =$ Closure
Identity
$r^{-1} = s$ ← Invertibility ⇒ Abelian Group.
$r \circ s = e$ ← Associativity
Commutativity

Express the transformations in matrix form and verify $\boxed{r \circ s = e.}$

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



2D rotation matrix → $R(\theta) = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

$$\Rightarrow r = \begin{vmatrix} \cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ \end{vmatrix} = \begin{vmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{vmatrix}$$

$$s = \begin{vmatrix} \cos 240^\circ & -\sin 240^\circ \\ \sin 240^\circ & \cos 240^\circ \end{vmatrix} = \begin{vmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{vmatrix}$$

$$r \circ s = \begin{vmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{vmatrix} \begin{vmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = e$$

S
-
ar
on
x
=

2) Prove that

$$S = \left(\left\{ \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \right\}, \times \right)$$

where $\times$ denotes matrix multiplication. is a matrix Group. Interpret the matrices as isometries and hence write down the corresponding isometry group.

Let : $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

reflection in line $y = -x$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $\leftarrow$ $a = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

reflection in line $y = x$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$c = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

rotate cclw $180°$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$a \times b = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = c$

$a \times c = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = b$

$b \times c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

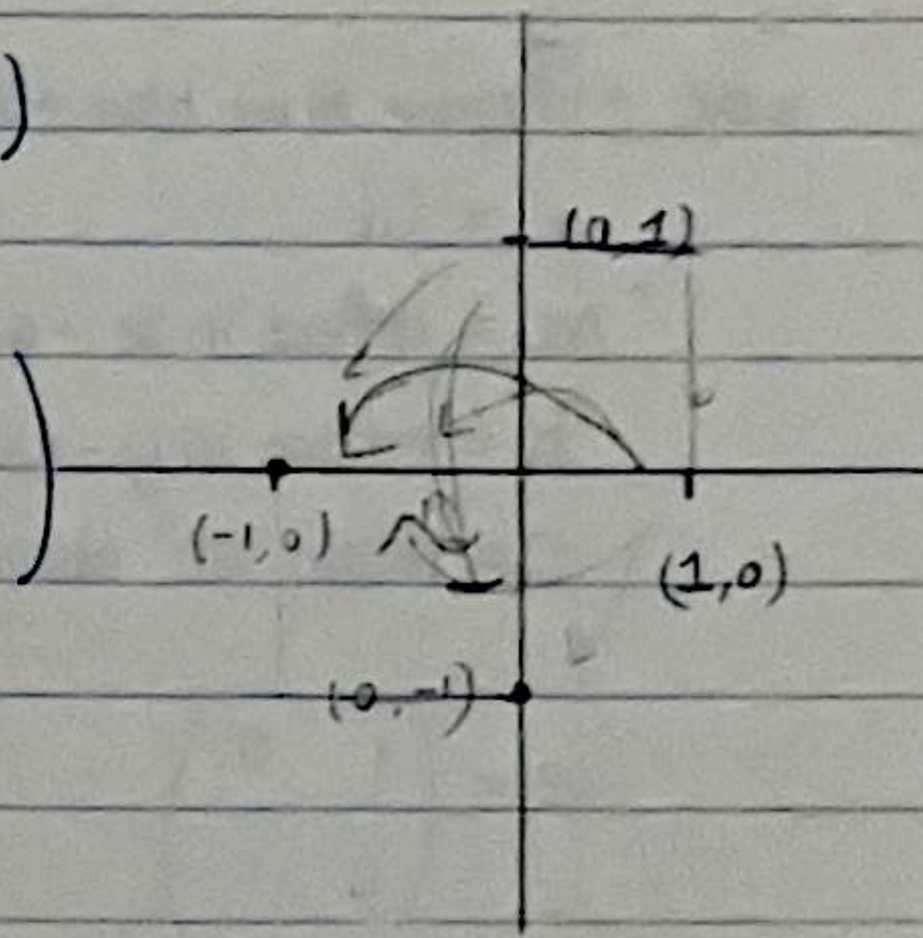
$= a$

$\Rightarrow$ Closure is satisfied

$b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow b^{-1} = \frac{1}{|b|} \cdot adj(b)$

$= \frac{1}{-1}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = b$



$\Rightarrow$ b is the inverse of itself

$\Rightarrow b \times b = e$ $\Rightarrow$ (Invertibility & Identity axioms are satisfied).

Associativity

$(a \times b) \times c = a \times (b \times c)$

$c \times c = a \times a$

$= e$

Since L.H.S = R.H.S $\Rightarrow$ Operation $\times$ is associative.

$b \times a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$= c$

$\Rightarrow$ Commutative

$\Rightarrow$ S is an abelian group.

3/ Prove that the transformations

$e = id$

a = reflect in x-axis

b = reflect in y-axis

r = rotate 180° about the origin.

Singular

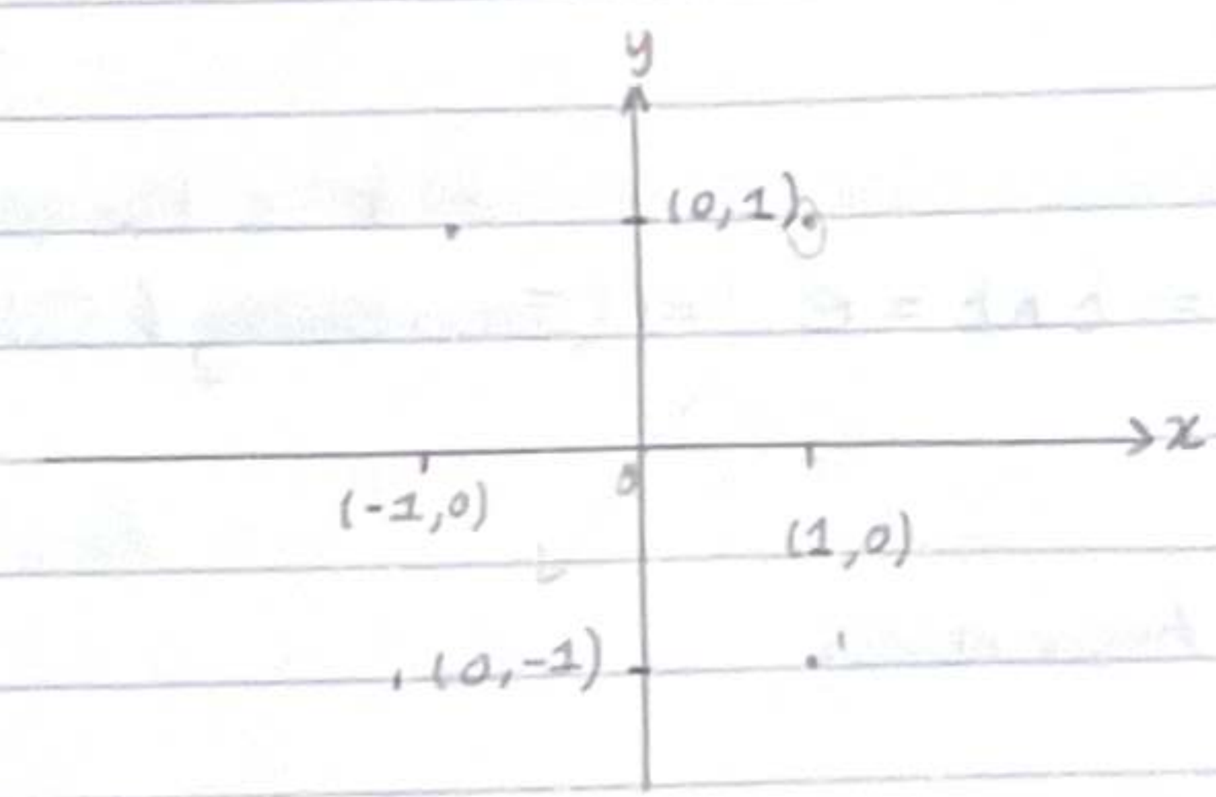
$S = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$|S| = |0| = 0$

det. $= 0 \Rightarrow$ Singularity

$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



$b = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$r = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

x	e	a	b	r
e	e	a	b	r
a	a	e	r	b
b	b	r	e	a
r	r	b	a	e

$\Rightarrow$ Closure
Associativity
Identity
Invertibility
Commutativity

$\Rightarrow$ Abelian Group

4) Prove that the set of matrices $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ where $r \in \mathbb{R}^+$ form a group under matrix multiplication.
Interpret this group geometrically.

Let $m, n \in \mathbb{R}^+$
s.t

1) Closure

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} = \begin{pmatrix} m+n & 0 \\ 0 & m+n \end{pmatrix}$$

If $m, n \in \mathbb{R}^+ \Rightarrow (m+n) \in \mathbb{R}^+$ (Well known property of +ve real numbers)

2) If $n = 1/m$

$$\Rightarrow \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e = id \text{ (Identity)}$$

& (Invertibility)

3) $$\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} = \begin{pmatrix} n+m & 0 \\ 0 & n+m \end{pmatrix}$$

$\Rightarrow \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \middle| r \in \mathbb{R}^+ \right\}, \times \}$ forms a group
where $\times$ denotes matrix multiplication.

$m + n = n + m$ (Addition is commutative on the reals) $\Rightarrow$ Abelian

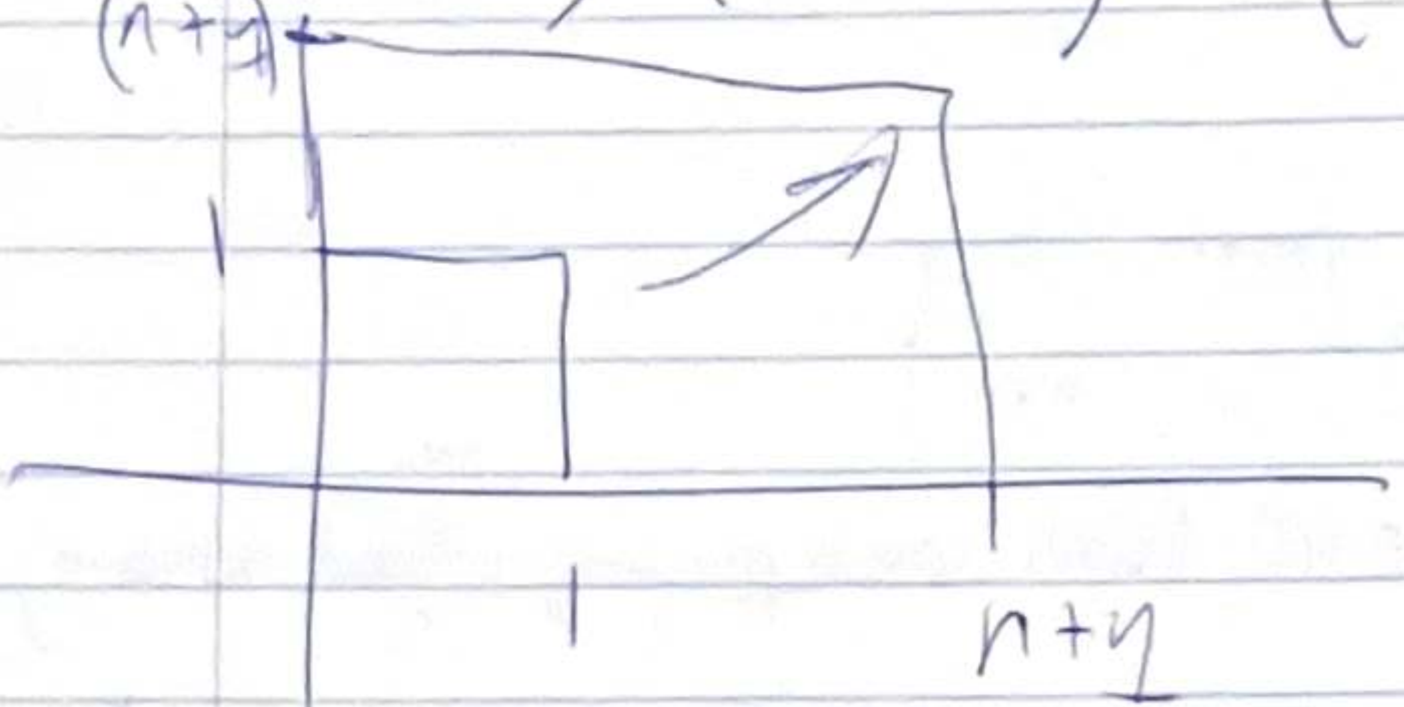
4) Associativity

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\left[\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}\right] = \left[\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}\right]\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} = \begin{pmatrix} m+n+j & 0 \\ 0 & m+n+j \end{pmatrix}$$

LHS = RHS $\Rightarrow$ Associative.

Geometric Interpretation

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} = \begin{pmatrix} n+1 & 0 \\ 0 & n+1 \end{pmatrix}$$



Enlarge by a factor of $\cancel{(n+1)}$

$$\frac{n+1}{n} = \left(1 + \frac{1}{n}\right)$$ ← factor of Enlargement

Enlargement (dilation by factor r about the origin.

Prove that the set of matrices of the form $\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$ where $r \in \mathbb{R}^+$ form a matrix group. Interpret it geometrically

Let $a, b, c \in \mathbb{R}^+$

1) $$\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 1/b \end{pmatrix} = \begin{pmatrix} a+b & 0 \\ 0 & \cancel{1/ab} \end{pmatrix} \quad \begin{pmatrix} a+b & 0 \\ 0 & 1/a + 1/b \end{pmatrix}$$

If $a, b \in \mathbb{R}^+ \Rightarrow (a+b)$ and $\left(\frac{1}{a} + \frac{1}{b}\right) \in \mathbb{R}^+$

$\Rightarrow$ Closure is satisfied.

3) $$\begin{pmatrix} b & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} = \begin{pmatrix} a+b & 0 \\ 0 & 1/a + 1/b \end{pmatrix}$$

$\Rightarrow$ Commutativity is Satisfied.

2) $A = \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$ $\quad A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj}(A)$

$$= \frac{1}{(1-0)} \cdot \begin{pmatrix} 1/r & 0 \\ 0 & r \end{pmatrix}$$

$$= \begin{pmatrix} 1/r & 0 \\ 0 & r \end{pmatrix}$$ where $r \in \mathbb{R}^+$

~~3,~~ 4) $\therefore AA^{-1} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\Rightarrow$ Invertibility and Identity are Satisfied

5) Associativity

$A(BC) \overset{!}{=} (AB)C$

$$\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}\begin{pmatrix} b+c & 0 \\ 0 & 1/b + 1/c \end{pmatrix} = \begin{pmatrix} a+b & 0 \\ 0 & 1/a + 1/b \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix}$$

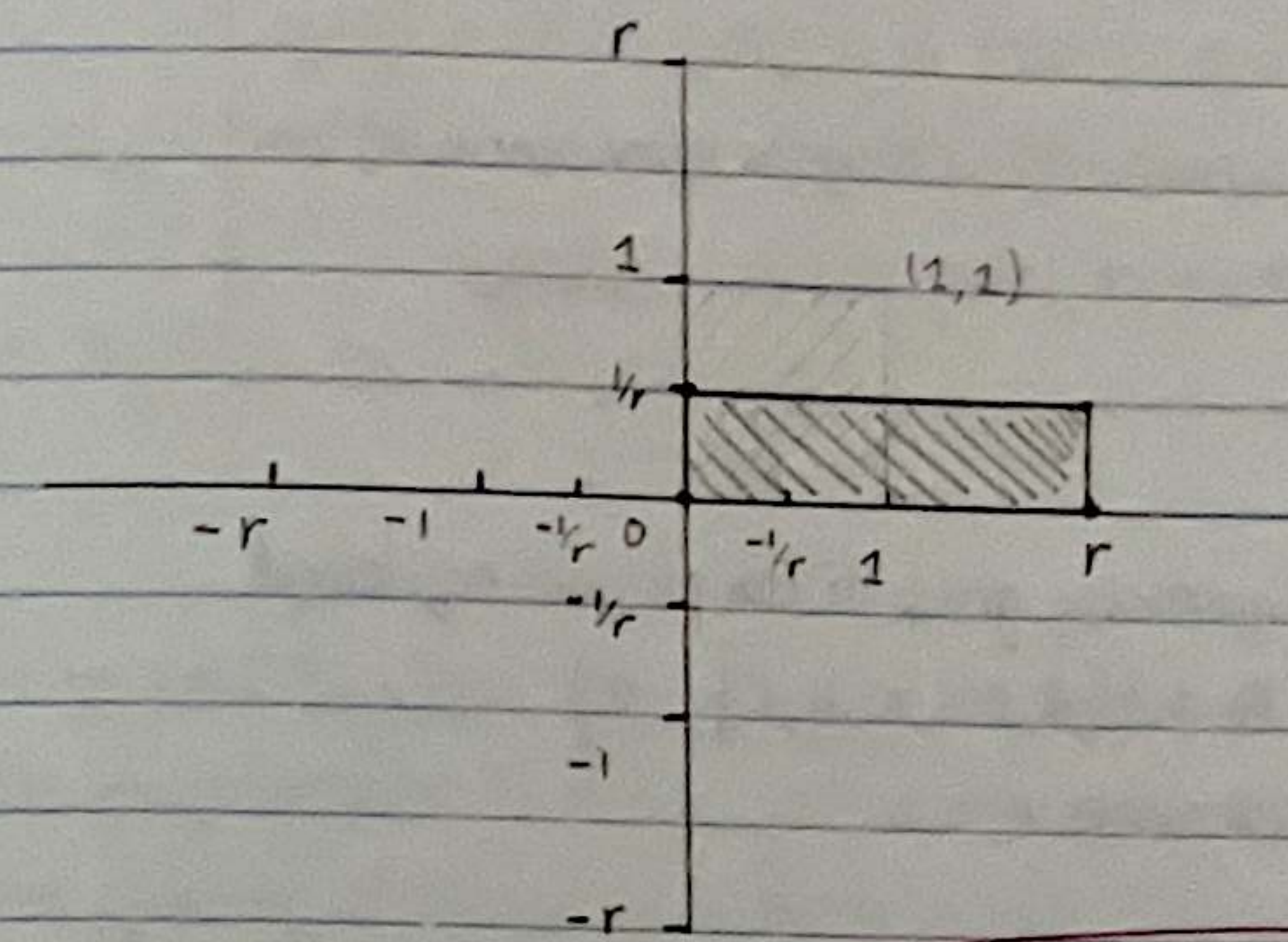
$$= \begin{pmatrix} a+b+c & 0 \\ 0 & 1/a + 1/b + 1/c \end{pmatrix}$$

Since LHS = RHS
$\Rightarrow$ Associativity is satisfied

$$\Rightarrow \left(\left\{ \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \middle| r \in \mathbb{R}^+ \right\}, \times \right)$$ where $\times$ denotes matrix multiplication

is an abelian group.

Geometric interpretation.



$$\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix}$$

This is a stretch with ~~stretc~~ stretch factor r.

$$\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/r \end{pmatrix}$$

Stretch x-axis and Shrinks y axis by same factor.

Subgroups

When a Subset of a group forms a group in its own right (using the same operation) then we say that the subset is a Subgroup.

Consider the Symmetry group of the previous equilateral triangle i.e. $G = (\{e,r,s,a,b,c\}, \circ)$
Consider the elements e, r, s.

$\circ$	e	r	s
e	e	r	s
r	r	s	e
s	s	e	r

We can see that $\{e,r,s\}$ form a group.

Axioms

1) The table shows closure

2) e is the identity element. $e \circ f = f \circ e = f$
e is in the subset and since e is the id for the whole symmetry group, then e is also the identity of the subset.

3) $r^{-1} = s$, $s^{-1} = r$, $s \circ r = e$
Inverse of all elements of the subset lie in the subset.

4) Since associativity holds for the whole group then it must hold for the subset.

In general, if $(H,*)$ is a subgroup of $(G,*)$ then we write $(H,*) \subseteq (G,*)$

↳ We need to check for 1) Closure
2) Invertibility
3) Identity.

① There is no need to check associativity as the latter is a property of the group operation. Since the group operation is defined on both the group and the subset then the associative property of the operation holds in both the group and its subset.

One-Step Subgroup test theorem

→ Let G be a group and let H be a non-empty subset of G. If for all a and b in H and ab^{-1} is in H then H is a subgroup of G.

Must inverse be in subset too?

That is for any group, a non-empty subset of that group is itself a group if the inverse of any element in the subset multiplied with any other element in the subset is also in the subset.

Declared in more rigorous 2-step subgroup test.

Reason ①

→ Notice: Identity, Closure and Associativity are not declared.

Inverse of a.

Consider the elements a and b which are in a subset of a group.
If a^{-1} exists then identity is proved trivially as $a^{-1} \circ a = e$

identity

Show that $(2\mathbb{Z},+)$ is a subgroup of $(\mathbb{Z},+)$ where $2\mathbb{Z}$ means the set of even integers.

~~Let $a,b \in \mathbb{Z}$ also $a,b \in 2\mathbb{Z}$~~

Let $a,b \in \mathbb{Z}$ also $a,b \in 2\mathbb{Z}$ ← Work on defn.

$a = 2m$

$a^{-1} = -2m$

$b = 2n$

$a^{-1} + b = 2n - 2m$

$= 2(n-m) \in 2\mathbb{Z}$

Via one-step subgroup test theorem $(2\mathbb{Z},+) \subseteq (\mathbb{Z},+)$

Longer Proof

- Closure

A typical element of $2\mathbb{Z}$ is $2n$ where $n \in \mathbb{Z}$

$2n + 2m = 2(n+m) \in 2\mathbb{Z} \longrightarrow$ The sum of even integers is even.

- Identity

0 is the identity in $(\mathbb{Z},+)$. Since $2 \times 0 = 0 \Rightarrow 0$ is also identity lies in the subset

- Invertibility

$(2n)^{-1} = -2n = 2(-n)$ which is also even.

2) Verify that $(\{e,a\},\circ)$ is a subgroup of the symmetry group of the equilateral triangle.

$\rightarrow (\{e,a\},\circ)$ is not a subgroup of D_3 as a has no inverse in the subset.

3) Prove that $(3\mathbb{Z},+) \subseteq (\mathbb{Z},+)$ where $3\mathbb{Z}$ is the set of all integers of form $3n$ (multiples of 3).

Let $a = 3n \in 3\mathbb{Z}$

$3\mathbb{Z} \in \mathbb{Z}$ (Property of integers)

$(3n)^{-1} = -3n$

$-3n \in 3\mathbb{Z}$

$\Rightarrow (3\mathbb{Z},+) \subseteq (\mathbb{Z},+)$ by One-step subgroup test theorem.

4) Verify that $\left(\left\{ \overset{e}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}, \overset{s}{\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}} \right\}, \times \right)$ is a subgroup of the matrix group in eg 14f

ccw 180° 90° 270°

1) $G = (\{e, s, r, t\}, \times)$

$H = (\{e, s\}, \times)$

$\cancel{S \circ S^{-1} = e}$

$\cancel{S^{-1} \circ S \circ S^{-1} = S^{-1} \circ e}$

If $S \circ K = e$

$\Rightarrow K = S^{-1}$

but if $K = S$

$\Rightarrow S \circ S = e$

$\Rightarrow$ S is the inverse of itself (involution)

$S \circ e = S \in H \quad (S = S^{-1})$

By One-step subgroup test theorem

$H(\{e, s\}, \times) \subseteq G(\{e, r, s, t\}, \times) \Leftarrow$

2) $M = \{$

2) $\cancel{M = \left(\left\{ \begin{vmatrix} 1 & n \\ 0 & 1 \end{vmatrix} \mid n \in \mathbb{Z} \right\}, \times\right)}$

$\cancel{H = \left(\left\{ \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \right\}, \times\right)}$

$\cancel{H \not\subseteq M \text{ as } \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \notin M} \Leftarrow$

$M = \left(\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \times\right)$

$H = \left(\left\{ \underset{e}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{A}{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}} \right\}, \times\right)$

$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj}(A)$

$\Rightarrow \boxed{H \subseteq M}$

$\cancel{= \frac{1}{1-0} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$

$= \frac{1}{(-1 \times -1) - (0 \times 0)} \cdot \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Leftarrow$ Involution.

4) Find all the Subgroup of the Symmetry group of the equilateral triangle. ccw 120° 240°

$D_3 = \{\{e, r, s, a, b, c\}, \circ\}$

$H_1 = (\{e, r, s\}, \circ)$

$H_2 = (\{e, a\}, \circ)$

$H_3 = (\{e, b\}, \circ)$

$H_4 = (\{e, c\}, \circ)$

$H_5 = (\{e\}, \circ)$

$H_6 = (\{e, r, s, a, b, c\}, \circ)$

5) Find all the Subgroups of

i) ~~$(Z_4, +4$~~ Addition modulo 4

$(Z_4, +_4)$

Integer modulo 4

$G = (\{0, 1, 2, 3\}, +4)$

~~H_1~~ =

$\forall p \in Z_4$

$p +_4 p \overset{!}{=} 0 \leftarrow$ Identity Element

$p +_4 (4-p) \overset{!}{=} 0$

$0 +_4 4 = 0 \leftarrow$

$\Rightarrow$ Inverse of p is $(4-p) \Leftarrow$

$\Rightarrow H_1 = (\{0, 1, 3\}, +_4)$ X why??

$H_2 = (\{0, 2\}, +_4)$

~~$H_3 = (\{0, 2, 4\}, +_4)$~~

$H_3 = (\{0, 1, 2, 3\}, +_4)$ or $(\{Z_4\}, +_4)$

$H_4 = (\{0\}, +_4)$

is $\boxed{H_1 \subseteq G\ ?}$

$H_1 = (\{0, 1, 3\}, +_4)$

$1 +_4 3 = 0$

$(1+3) \bmod 4$

$= 4 \bmod 4 = 0$

but 4 is not in the group G.

$\Rightarrow H_1 \not\subseteq G$

??

~~ii) $\mathbb{Z}/6\mathbb{Z} = (\mathbb{Z}_6, +_6)$~~

~~$= (\{0,1,2,3,4,5\}, +_6)$~~

~~$H_1 = (\{0\}, +_6)$~~

~~$H_2 = ($~~

Find all the subgroups of $(\mathbb{Z}_4, +_4)$ $\boxed{\text{or } \mathbb{Z}/4\mathbb{Z}}$

$\cancel{G} = \mathbb{Z}/4\mathbb{Z} = (\{0,1,2,3\}, +_4)$

$H_1 = (\{0\}, +_4)$

$H_2 = (\{0,1,2,3\}, +_4)$ or $(\mathbb{Z}_4, +_4)$

$H_3 = (\{0,2\}, +_4)$

Why is $(\{0,1,3\} +_4)$ not a subgroup?

$+_4$	0	1	3
0	0	1	3
1	1	2	0
3	3	0	2

$(\{0,1,3\}, +_4)$ is not a subgroup of $\mathbb{Z}/4\mathbb{Z}$ as 2 is present ie. no closure.

2 step subgroup test theorem.

$(\{0,1,2\}, +_4)$ is also not a subgroup of $\mathbb{Z}/4\mathbb{Z}$

as 3 is present in the cayley table.

2 & 3 elements

ii) ~~Is $(\mathbb{Z}_6, +_6)$ a Sub~~
List all the subgroups of:
$G = (\mathbb{Z}_6, +_6) = (\{0,1,2,3,4,5\}, +6)$

	0	3
0	0	3
3	3	0

Trivial Subgroups $\begin{cases} H_1 = (\{0\}, +6) \\ H_2 = (\mathbb{Z}_6, +_6) \end{cases}$

~~$H_3 = (\{0,1\}, +_6)$~~

	0	2	4
0	0	2	4
2	2	4	0
4	4	0	2

$H_3 = (\{0,3\}, +_6)$

$H_4 = (\{0,2,4\}, +_6)$

$(0, 1, 2, 5)$

~~Addit~~

Subgroups of the Additive group of Integers modulo n.

$\mathbb{Z}/_{n\mathbb{Z}} = (\{0,1,2,3,\dots,(n-1)\}, +_n)$

Even - n

Consider $\mathbb{Z}/_{4\mathbb{Z}}$ — Cardinality or $n(\mathbb{Z}_4) = 4$

Cardinality of Powerset of $\mathbb{Z}_4 = 2^4 = 16$

$\mathbb{Z}/_{4\mathbb{Z}} = (\{0,1,2,3\}, +_4)$

Power set of $\mathbb{Z}_4$ $P(\mathbb{Z}_4)$

$= \{0\}$ ✓ $\quad \{\}$ ✗ $\quad \{2,3\}$

$\{0,1\} \quad \{0,1,2,3\}$ ✓ $\quad \{1,2,3\}$

$\{0,2\} \quad \{0,1,2\}$

$\{0,3\} \quad \{0,1,3\}$

$\{1\} \quad \{0,2,3\}$

$\{2\} \quad \{1,2\}$

$\{3\} \quad \{1,3\}$

① Ignore null - set
② Account for $\{0\}$ and $\{0,1,2,3\}$
We need

7) Consider the dihedral group D_n

g is a reflection

ie $g: x \mapsto x'$
and $g^{-1}: x' \mapsto x$ } reflection g is an involution

$\therefore g = g^{-1}$

In the Subset, there is g and e

$g^{-1} \circ e \in$ Subset

$g^{-1} \circ e = g \circ e \in \{g, e\}$

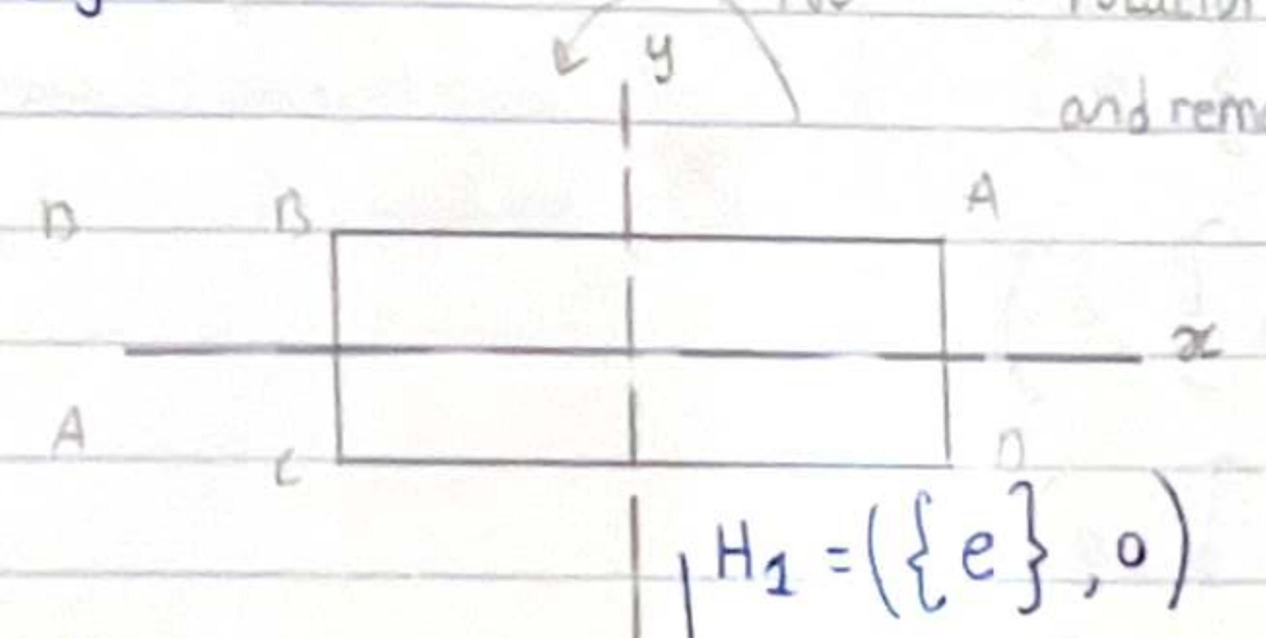
By One-step subgroup test theorem

$(\{g,e\}, \circ) \subseteq D_n$ Since $g^{-1} \circ e$ and $g^{-1} \circ g \in \{g,e\}$

8) Find all the Subgroups of the Symmetry groups of

i) the rectangle.

$r = 180°$ ⟶ rotation which preserves rigidity and remain in same orientation in space.



$G = (\{e, r, x, y\}, \circ)$

$H_1 = (\{e\}, \circ)$

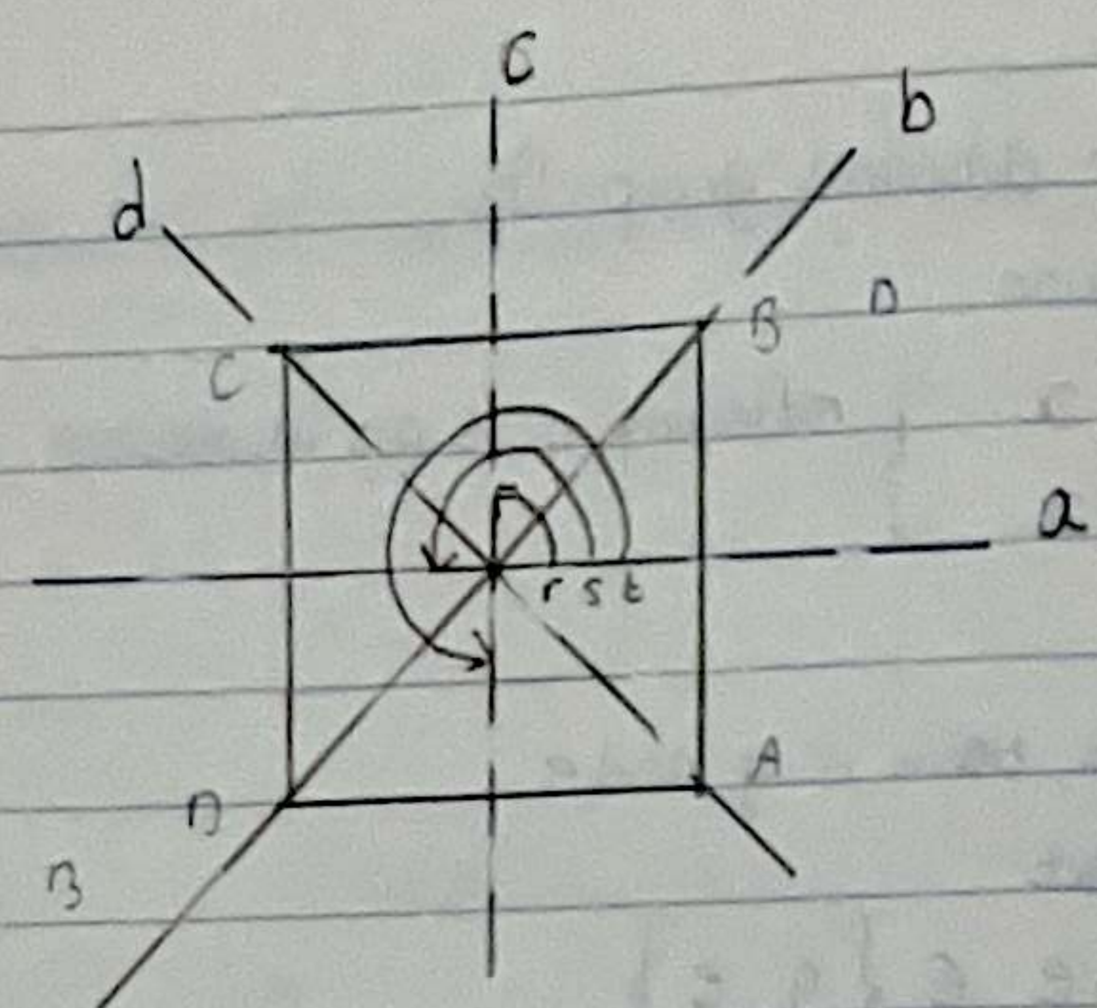
$H_2 = (\{e, r, x, y\}, \circ)$

$H_3 = (\{e, r\}, \circ)$

$H_4 = (\{e, x\}, \circ)$

$H_5 = (\{e, y\}, \circ)$

ii) The Square.



8

4 2

$$D_4 = (\{e, a, b, c, d, r, s, t\}, \circ)$$

$$H_1 = (\{e\}, \circ)$$

$$H_2 = D_4$$

$$H_3 = (\{e, a\}, \circ)$$

$$H_2 = (\{e, b\}, \circ)$$

$$H_3 = (\{e, c\}, \circ)$$

$$H_4 = (\{e, d\}, \circ)$$

$$H_5 = (\{e, s\}, \circ)$$

$$H_6 = (\{e, r, s, t\}, \circ)$$

Review

Two more in answers in book.

Isomorphism

Augmented matrix

Gauss - Jordan Method & Gaussian Elimination

Gaussian Elimination

$$\left.\begin{aligned} a_1 x + b_1 y + c_1 z &= k_1 \\ a_2 x + b_2 y + c_2 z &= k_2 \\ a_3 x + b_3 y + c_3 z &= k_3 \end{aligned}\right\} \longrightarrow \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & k_1 \\ a_2 & b_2 & c_2 & k_2 \\ a_3 & b_3 & c_3 & k_3 \end{array}\right]$$

Challenge: Put a matrix in row-eschelon form after Gaussian Elimination has operated on the rows.

ie. $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & k_1 \\ 0 & b_2 & c_2 & k_2 \\ 0 & 0 & c_3 & k_3 \end{array}\right]$

$\longrightarrow$ mapping a structre to another.

$\longmapsto$ Send element of a structre to another

Isomorphism

$(G, \circ)$ $(H, *)$

Given 2 groups ~~$G(S_1, \circ)$~~ and ~~$H(S_2, *)$~~, a group isomorphism from G to H is a bijective group homomorphism from G to H.

Both injective & Surjective

One ~~object~~ element in domain has only one corresponding element in range

No elements in range are left unpaired.

A group homomorphism from $(G, \circ)$ to $(H, *)$ is a function $h : G \mapsto H$ such that

~~$\forall u, v \in G$~~

~~$h(u \circ v) = h(u)$~~

$\forall u, v \in G$

$$h(u \circ v) = h(u) * h(v)$$

Send v from G to H

Compose

Group operation in G

u & v in G then Send to H via h

Send u from G to H

Group operation in H.

G

$\circ$	u	v		
u		$u \circ v$		
v				

h

Note if G is homomorphic to H then $h(u) = u'$
$h(v) = v'$

H $*$	u'	v'	
u'		$u' * v'$	
v'			

$= h(u) * h(v)$

From this property We can See that h maps the identity of G to the identity of H

ie $h(e_G) = e_H$

$\phi : (G, \circ) \to (H, *)$

$\phi(e_G) = \phi(e_G \circ e_G) = \phi(e_G) * \phi(e_G)$

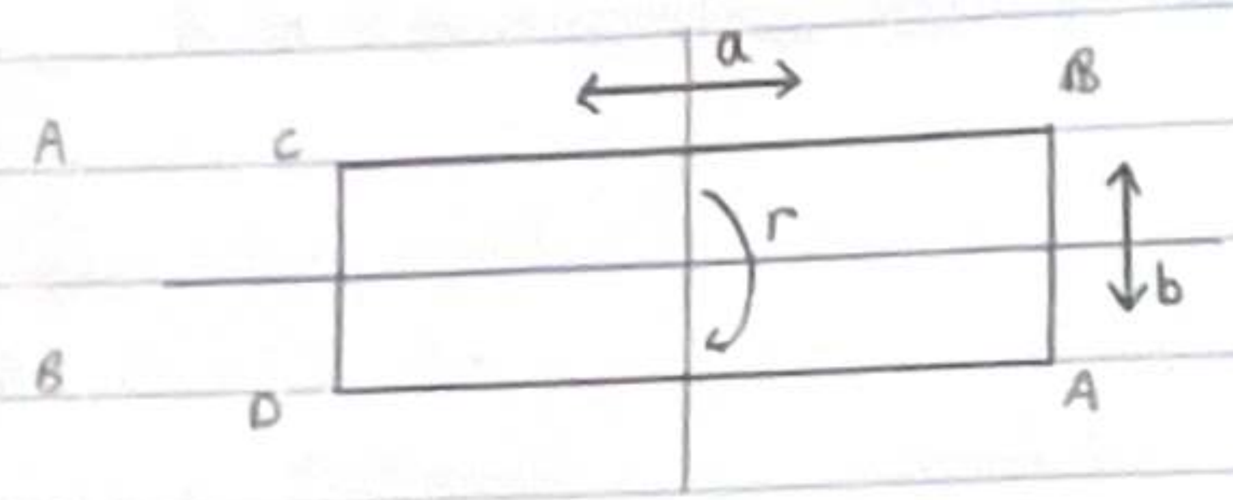
$\phi(e_G) * e_H = \phi(e_G) * \phi(e_G)$

$e_H = \phi(e_G)$

and it also maps the inverse to some inverses

$h(u^{-1}) = h(u)^{-1}$

Consider the Symmetry Group of the rectangle.



$\circ$	e	a	b	r
e	e	a	b	r
a	a	e	r	b
b	b	r	e	a
r	r	b	a	e

Consider the matrix group $\left(\left\{\begin{vmatrix}1&0\\0&1\end{vmatrix}, \begin{vmatrix}0&1\\1&0\end{vmatrix}, \begin{pmatrix}0&-1\\-1&0\end{pmatrix}, \begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right\}, X\right)$
(from previous exercise)

X	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$
$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$
$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$
$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$

Notice : Consider the following correspondence between the elements of the 2 groups.

$e \leftrightarrow \begin{pmatrix}1&0\\0&1\end{pmatrix}$ $\quad a \leftrightarrow \begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\quad b \leftrightarrow \begin{pmatrix}0&-1\\-1&0\end{pmatrix}$ $\quad r \leftrightarrow \begin{pmatrix}-1&0\\0&-1\end{pmatrix}$

Bijectivity denotes isomorphism (and not homomorphism)

It can be seen that the elements of the second table correspond to those of the first table but are just relabelled. Either table can be used to work out the composition of elements in either groups.

We say that the 2 groups are isomorphic and is denoted as follows

$$\underbrace{(\{e,a,b,r\},\circ)}_{(G,*)} \cong \underbrace{\left(\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-1\\-1&0\end{pmatrix}, \begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right\}, \times\right)}_{(H,\square)}$$

Generally, Groups G and H are isomorphic if it is possible to set up a one-to-one correspondence between them. (There exists a bijective homomorphism between groups G and H.)

ie. $\left.\begin{matrix} g \leftrightarrow g' \\ h \leftrightarrow h' \end{matrix}\right\} \Rightarrow g * h \leftrightarrow g' \square h'$

We write $(G,*) \cong (H,\square)$

Ex. Prove that the Symmetry group of the equilateral triangle is isomorphic to the permutation group $(S_3, \circ)$ on the set $\{1,2,3\}$

№ Symbols = 3

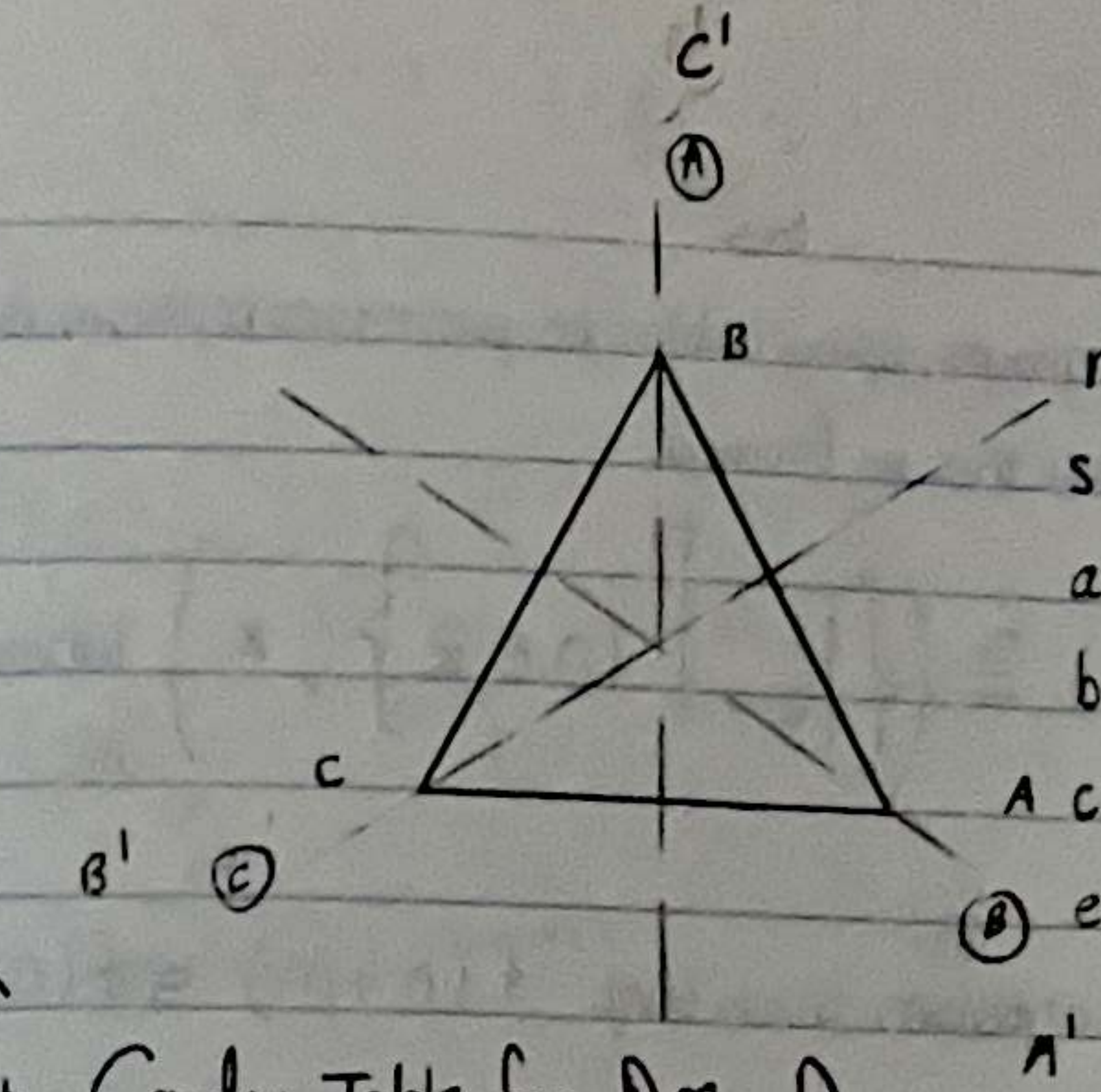
№ possible permutations = $3! = 6$

For 3 elements keep 1 fixed then transpose then iterate.

1 2 3
1 3 2
2 1 3
2 3 1
3 1 2
3 2 1

Cayley Table for $(S_3, \circ)$

	$\circ$	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$
e	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$
reflection	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$
	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$
	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$
rotation	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&2&3\end{pmatrix}$
	$\begin{pmatrix}1&2&3\\3&2&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$



$r = 120^\circ$ cc/w
$s = 240^\circ$ cc/w
a = reflect thru' A
b = reflect thru' B
c = reflect thru' C
e = id.

Not Commutative

Order of operation must be respected

~~Symmet~~ Cayley Table for ~~D~~ D_3

$\circ$	e	a	b	c	r	s
e	e	a	b	c	r	s
a	a	e	r	r	c	b
b	b	r	e	s	a	c
c	c	r	s	e	b	a
r	r	b	c	a	s	e
s	s	c	a	b	e	r

Note: Even if the Cayley Tables are not ordered, isomorphism still holds as it represents the mapping of composition of elements from one table to another and not their place in the table.

When the Cayley Tables are reordered, the pattern can be better seen.

Notice

$e \longleftrightarrow \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$ $a \longleftrightarrow \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$ $b \longleftrightarrow \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$

$c \longleftrightarrow \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$ $r \longleftrightarrow \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$ $s \longleftrightarrow \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$

Formal representation of isomorphism function.

$\Rightarrow (S_3, \circ) \cong D_3$

Symmetry (Permutation) group on 3 symbols.

Dihedral group degree 3 (order 6)

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} + \begin{pmatrix} E & G \\ F & H \end{pmatrix} = \begin{pmatrix} A+E & C+G \\ B+F & D+H \end{pmatrix}$$

2) In cases of infinite groups, using tables to prove isomorphism is not practical. A general argument must thus be formed.

eg. Prove that $\underset{G}{(\mathbb{Z},+)} \cong \underset{H}{\left(\left\{\begin{vmatrix} 1 & n \\ 0 & 1 \end{vmatrix} \;\middle|\; n \in \mathbb{Z}\right\}, \times\right)}$ where $\times$ denotes matrix multiplication.

Let f be an isomorphism such that $f(n+m) = f(n) \times f(m)$
$f : G \leftrightarrow H$

$f(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = id$ $\quad$ ~~$f : n \leftrightarrow \begin{vmatrix} 1 & n \\ 0 & 1 \end{vmatrix}$~~

~~Here we add the matrices~~

$f(n) = \begin{vmatrix} 1 & n \\ 0 & 1 \end{vmatrix}$ $\quad \Longrightarrow \quad f(n+m) = \begin{vmatrix} 1 & n+m \\ 0 & 1 \end{vmatrix}$

$f(m) = \begin{vmatrix} 1 & m \\ 0 & 1 \end{vmatrix}$ $\quad$ ~~$f : m \leftrightarrow \begin{vmatrix} 1 & m \\ 0 & 1 \end{vmatrix}$~~

No: we do not add the matrices but define a new argument of f ie. $(n+m)$

~~$f(n+m)$~~

$f(n) \times f(m) = \begin{vmatrix} 1 & n \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & m \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & n+m \\ 0 & 1 \end{vmatrix}$

~~Here we multiply the matrices.~~

$\Rightarrow f(n+m) = f(n) \times f(m)$

$\Rightarrow G \cong H$

Let f be an isomorphism $f : G \leftrightarrow H$

$\Rightarrow f(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

$f(m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$

$f(n+m) = \begin{pmatrix} 1 & n+m \\ 0 & 1 \end{pmatrix}$

Now we evaluate $f(n) \times f(m)$

$= \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+m \\ 0 & 1 \end{pmatrix}$

$\Rightarrow f(n+m) = f(n) \times f(m)$

$\Rightarrow G \cong H$

or If $n \leftrightarrow \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, $m \leftrightarrow \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ $\Big\}$ Obvious Correspondence $n, m \in \mathbb{Z}$

then $(n+m)$ in $(\mathbb{Z},+)$ and $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ in matrix [illegible] are the same

$\Rightarrow n+m \leftrightarrow \begin{pmatrix} 1 & n+m \\ 0 & 1 \end{pmatrix}$

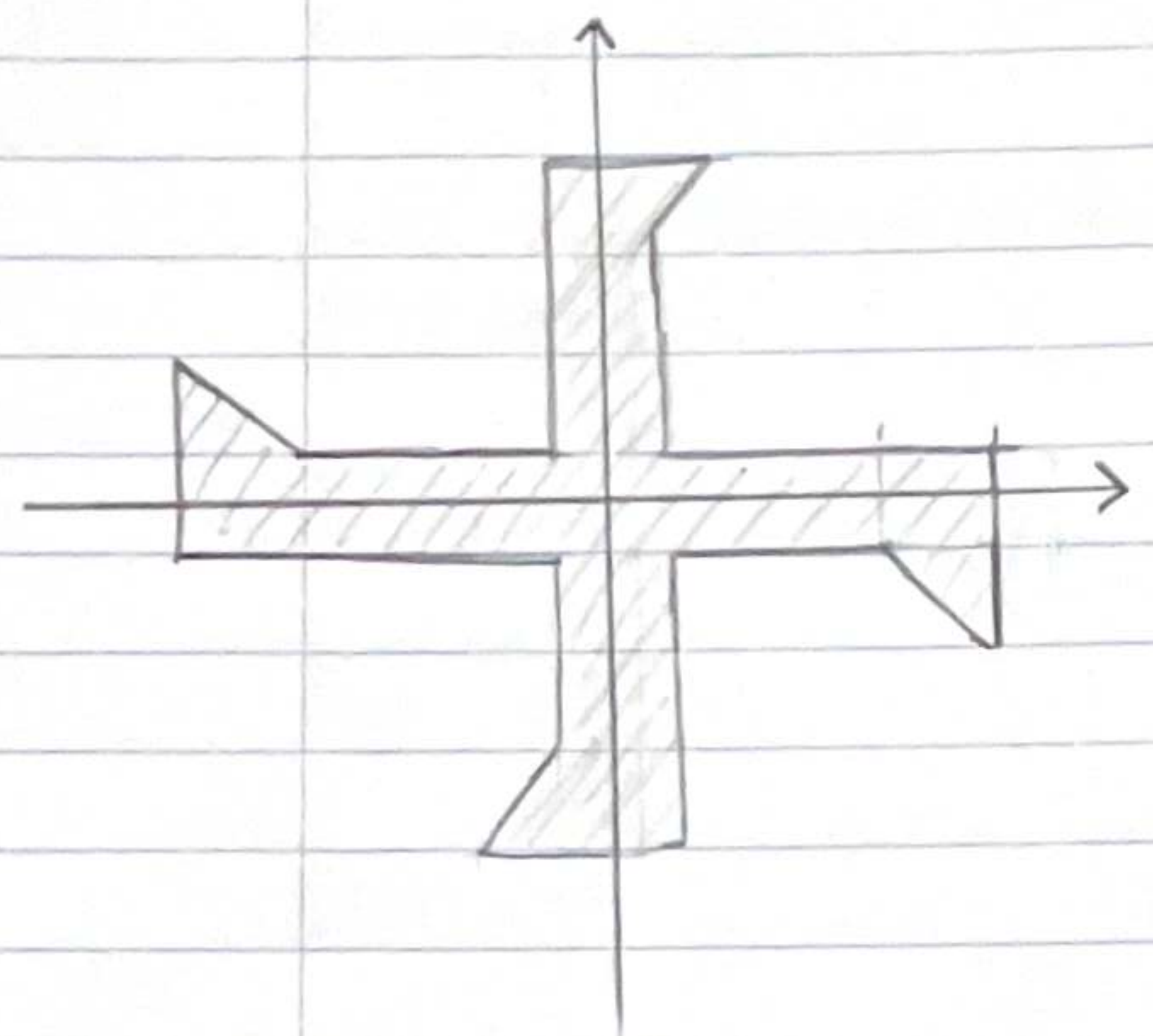
PBI

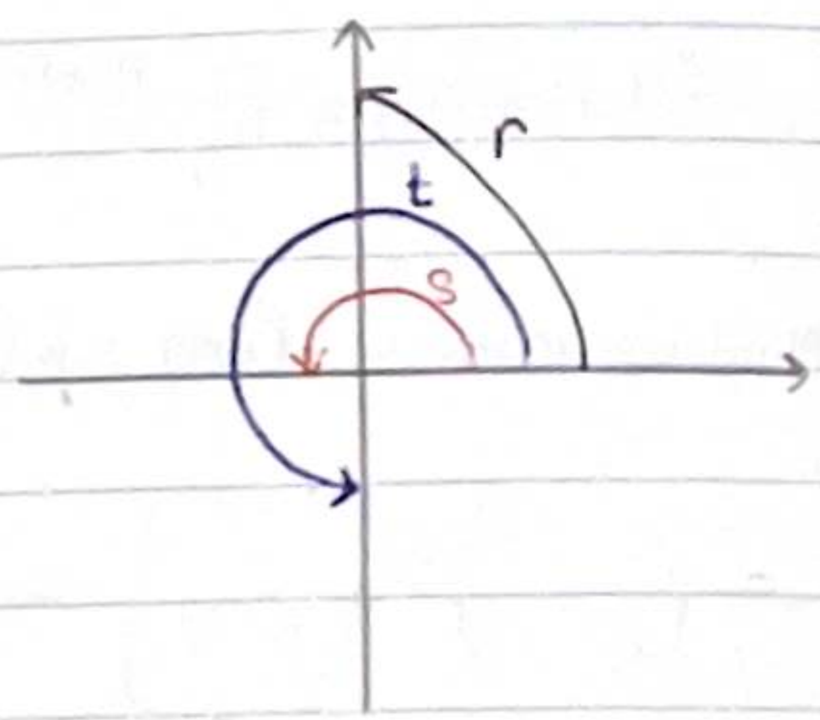
Argument using labels

$(\{e,r,s,t\}, \circ)$

Geometric Arguments to prove isomorphism

Prove that the isometry group of the rotation of the plane by 90° is isomorphic to the symmetry group of the 4-blade windmill.





- The 4-blade windmill is positioned with its centre at the origin.
- The windmill has only rotational symmetry (no reflection).
- The 4 isometries of the plane $\{e,r,s,t\}$ then actually realise the 4 symmetries of the windmill.

$\Rightarrow$ The composition of isometries correspond to the composition of symmetries

$\Rightarrow$ The groups are isomorphic.

Not rigorous enough.

What if windmill has more than 4 symmetries?

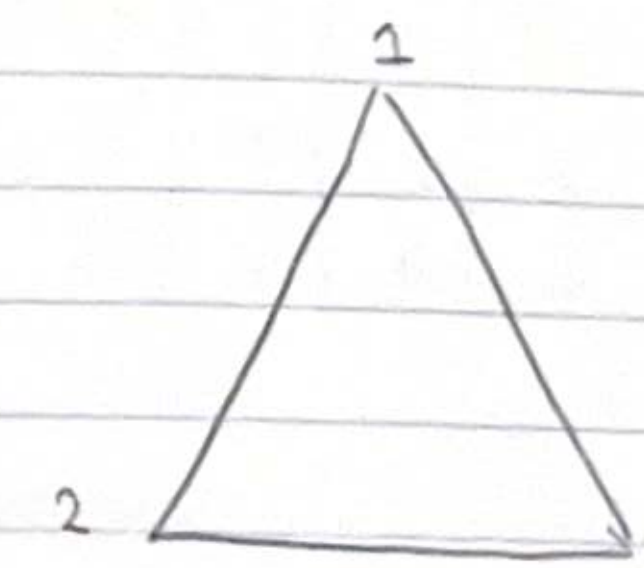
Use a "labelling" argument to establish the isomorphism between the symmetry group of the equilateral triangle and $(S_3, \circ)$

Permutation group on 3 symbols under composition.

We use the labels 1, 2, 3 for the $\underline{\text{posn}}$ of the corners of the $\triangle$ (Not A, B, C).

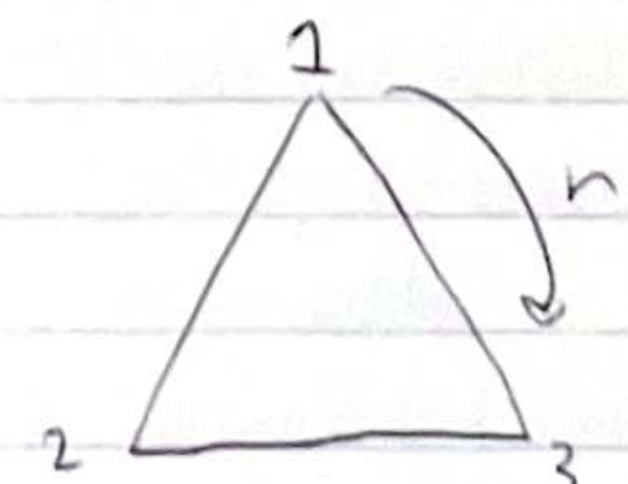
(labelling) This can be done as we already proved that the 2 groups satisfy the group axioms

we only need to show order is same & commutativity.

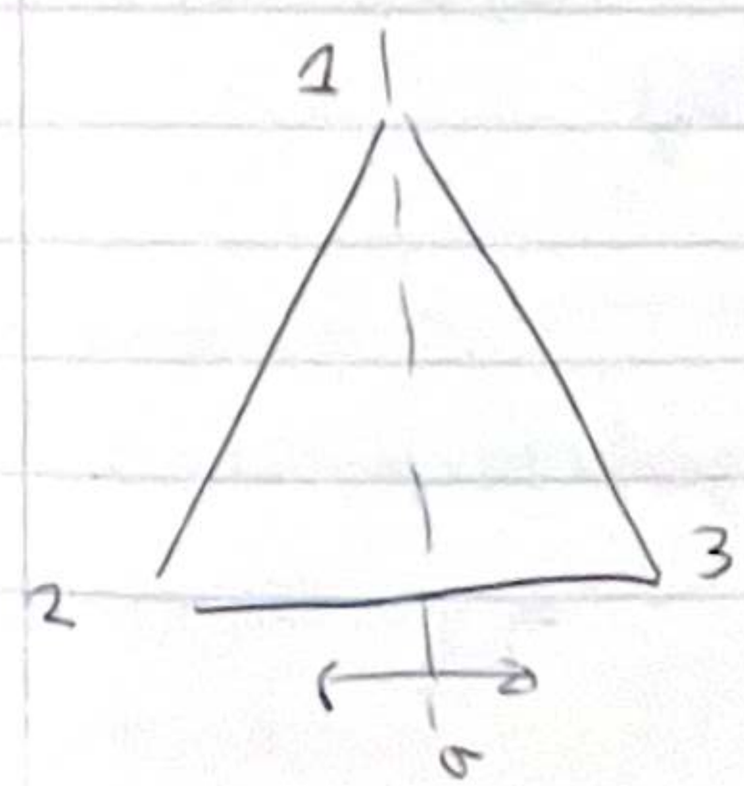


$\therefore$ Each symmetry of the $\triangle$ determines a permutation of $\{1,2,3\}$ where each corner i is mapped to another corner j under the symmetry.

eg.



gives $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$



gives $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

Now the composition of the 2 symmetries has for effect the composition of the permutations one after the other.

$\Rightarrow D_3 \cong (S_3, \circ)$

More general Arguments

(still use labels)