14h/ Use group tables to establish the following group isomorphisms.

(a) $(\mathbb{Z}_3, +_3) \cong (\{0,2,4\}, +_6)$

For $(\mathbb{Z}_3, +_3)$

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

For $(\{0,2,4\}, +_6)$

$+_6$	0	2	4
0	0	2	4
2	2	4	0
4	4	0	2

Via inspection
isomorphism ~~f: 0↔~~
$f : \underset{G}{(\mathbb{Z}_3, +_3)} \leftrightarrow \underset{H}{(\{0,2,4\}, +_6)}$ or $f: (\mathbb{Z}_3, +_3) \mapsto (\{0,2,4\}, +_6)$
where f is bijective.

G H
$0 \leftrightarrow 0$
$1 \leftrightarrow 2$
$2 \leftrightarrow 4$

$\Rightarrow G \cong H$

MORE RIGOROUS Suppose $f(0 +_3 0) = a$ (look for this in table for H)
and element in H $f(0)$ composed with itself yields b ie $f(0) +_6 f(0) = b$??
$f : G \to H$
$f(0 +_3 0) = f(0) +_6 f(0)$ $\Rightarrow f(0) = 0$ From $f(0 +_3 0) = f(0) +_6 f(0)$ we get $a = b$ in G in H.
We see 0 is the identity element in both G and H.
This leave only 2 possibilities for $f(1)$ and $f(2)$
Now $f(1 +_3 0) = f(1) +_6 f(0)$ with $f(0) = 0$
from G we get 1

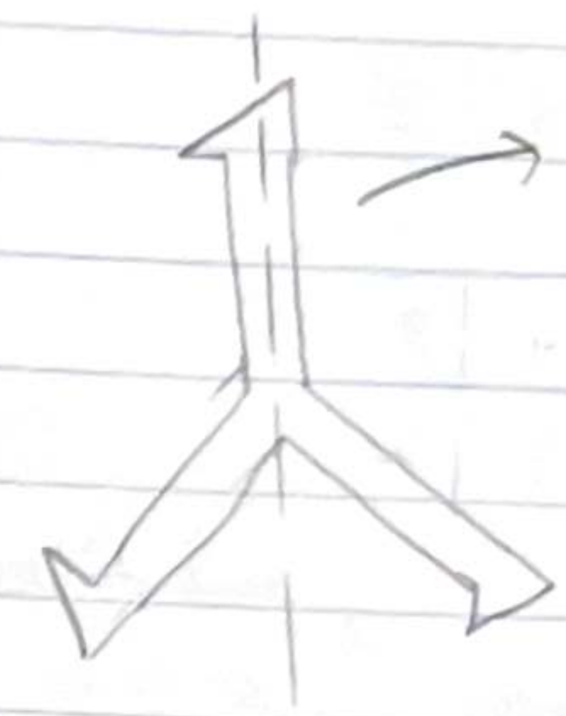
$1 \to f(1)$ 1 can be equivalent to either 2 or 4 !!
$2 \to f(2)$
$f(1+2) \to f(1) +_6 f(2)$, $1+2 = 0$
?? How do we know?!

(b) $\underset{G}{(\mathbb{Z}_3, +_3)} \cong \underset{H}{\text{Symmetry group of 3-bladed windmill.}}$

For $(\mathbb{Z}_3, +_3)$

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

For 3-bladed Windmill.



Has no reflectional Symmetry
$r = 120°$ cc/w
$s = 240°$ cc/w
$e = id.$

Via Inspection,
There is an isomorphism
$f : G \to H$ (where f is ~~big~~ bijective)
Isomorphism and NOT Homomorphism

o	e	r	s
e	e	r	s
r	r	s	e
s	s	e	r

$f(n +_3 m) = f(n) \circ f(m)$??
or $f(n +_3 m) = f(m) \circ f(n)$ ← Do n first according to G. ??

G H
$f: 0 \leftrightarrow e$
$1 \leftrightarrow r$
$2 \leftrightarrow s$

$\Rightarrow G \cong H.$

$$\overset{G}{(c)\ (\{e,r,s\},\circ)} \cong H$$

where H is

$*$	x	y	z
x	z	x	y
y	x	y	z
z	y	z	x

G

$\circ$	e	r	s
e	e	r	s
r	r	s	e
s	s	e	r

Study the relations between elements in each group, NOT their positions.

via inspection

~~$x \to r$~~

~~$y \to$~~

$y \leftrightarrow e$
$x \leftrightarrow s$
$z \leftrightarrow r$

$\longrightarrow$ Make some sort of simultaneous equations.

(d) Show $(\mathbb{Z}_4, +_4)$ is isomorphic to the rotation of the plane 90° cc/w.

For $(\mathbb{Z}_4, +_4)$ G

$\searrow$ $e = id$

$r = 90^\circ$ cc/w

$s = 180^\circ$ cc/w

$t = 270^\circ$ cc/w

$$\begin{array}{r} 4\ 1\ 0 \\ -\ \cancel{5}\cancel{4}0 \\ 360 \\ \hline 180 \end{array}$$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

For plane rotation. H

$\circ$	e	r	s	t
e	e	r	s	t
r	r	s	t	e
s	s	t	e	r
t	t	e	r	s

There exists an isomorphism f s.t $f: G \to H$ (where f is bijective)

$f(x +_4 y) = f(x) \circ f(y)$

$0 \leftrightarrow e$

$1 \leftrightarrow r$

$2 \leftrightarrow s$

$3 \leftrightarrow t$

$$G$$

(e) $\left(\left\{\begin{vmatrix}1&0\\0&1\end{vmatrix}, \begin{vmatrix}0&1\\1&0\end{vmatrix}, \begin{vmatrix}0&-1\\-1&0\end{vmatrix}, \begin{vmatrix}-1&0\\0&-1\end{vmatrix}\right\}, X\right)$

$\cong \left(\left\{\begin{vmatrix}1&0\\0&1\end{vmatrix}, \begin{vmatrix}-1&0\\0&1\end{vmatrix}, \begin{vmatrix}1&0\\0&-1\end{vmatrix}, \begin{vmatrix}-1&0\\0&-1\end{vmatrix}\right\}, X\right)$

$$H$$

G

X	e $\begin{vmatrix}1&0\\0&1\end{vmatrix}$	a $\begin{vmatrix}0&1\\1&0\end{vmatrix}$	b $\begin{vmatrix}0&-1\\-1&0\end{vmatrix}$	c $\begin{vmatrix}-1&0\\0&-1\end{vmatrix}$
e $\begin{vmatrix}1&0\\0&1\end{vmatrix}$	e $\begin{vmatrix}1&0\\0&1\end{vmatrix}$	a $\begin{vmatrix}0&1\\1&0\end{vmatrix}$	b $\begin{vmatrix}0&-1\\-1&0\end{vmatrix}$	c $\begin{vmatrix}-1&0\\0&-1\end{vmatrix}$
a $\begin{vmatrix}0&1\\1&0\end{vmatrix}$	a $\begin{vmatrix}0&1\\1&0\end{vmatrix}$	e $\begin{vmatrix}1&0\\0&1\end{vmatrix}$	c $\begin{vmatrix}-1&0\\0&-1\end{vmatrix}$	b $\begin{vmatrix}0&-1\\-1&0\end{vmatrix}$
b $\begin{vmatrix}0&-1\\-1&0\end{vmatrix}$	b $\begin{vmatrix}0&-1\\-1&0\end{vmatrix}$	c $\begin{vmatrix}-1&0\\0&-1\end{vmatrix}$	e $\begin{vmatrix}1&0\\0&1\end{vmatrix}$	a $\begin{vmatrix}0&1\\1&0\end{vmatrix}$
c $\begin{vmatrix}-1&0\\0&-1\end{vmatrix}$	c $\begin{vmatrix}-1&0\\0&-1\end{vmatrix}$	b $\begin{vmatrix}0&-1\\-1&0\end{vmatrix}$	a $\begin{vmatrix}0&1\\1&0\end{vmatrix}$	e $\begin{vmatrix}1&0\\0&1\end{vmatrix}$

row • column

$\begin{vmatrix}0&-1\\-1&0\end{vmatrix}\begin{vmatrix}0&-1\\-1&0\end{vmatrix} = \begin{vmatrix}1&0\\0&1\end{vmatrix}$

$\begin{vmatrix}0&1\\1&0\end{vmatrix}\begin{vmatrix}0&1\\1&0\end{vmatrix} = \begin{vmatrix}1&0\\0&1\end{vmatrix}$ $A = \begin{pmatrix}a&c\\b&d\end{pmatrix}$ $A' = \begin{pmatrix}ka&kc\\kb&kd\end{pmatrix} = kA$

$\begin{vmatrix}0&1\\1&0\end{vmatrix}\begin{vmatrix}0&-1\\-1&0\end{vmatrix} = \begin{vmatrix}-1&0\\0&-1\end{vmatrix}$ $B = \begin{pmatrix}e&g\\f&h\end{pmatrix}$ $\Rightarrow k(AB)$

$\begin{vmatrix}0&1\\1&0\end{vmatrix}\begin{vmatrix}-1&0\\0&-1\end{vmatrix} = \begin{vmatrix}0&-1\\-1&0\end{vmatrix}$ $AB = \begin{pmatrix}\cdots\\\cdots\end{pmatrix}$

$\begin{vmatrix}0&-1\\-1&0\end{vmatrix}\begin{vmatrix}-1&0\\0&-1\end{vmatrix} = \begin{vmatrix}0&1\\1&0\end{vmatrix}$

H

X	e $\begin{vmatrix}1&0\\0&1\end{vmatrix}$	f $\begin{vmatrix}-1&0\\0&1\end{vmatrix}$	g $\begin{vmatrix}1&0\\0&-1\end{vmatrix}$	h $\begin{vmatrix}-1&0\\0&-1\end{vmatrix}$
e $\begin{vmatrix}1&0\\0&1\end{vmatrix}$	e $\begin{vmatrix}1&0\\0&1\end{vmatrix}$	f $\begin{vmatrix}-1&0\\0&1\end{vmatrix}$	g $\begin{vmatrix}1&0\\0&-1\end{vmatrix}$	h $\begin{vmatrix}-1&0\\0&-1\end{vmatrix}$
f $\begin{vmatrix}-1&0\\0&1\end{vmatrix}$	f $\begin{vmatrix}-1&0\\0&-1\end{vmatrix}$	e $\begin{vmatrix}1&0\\0&1\end{vmatrix}$	h $\begin{vmatrix}-1&0\\0&-1\end{vmatrix}$	g $\begin{vmatrix}1&0\\0&-1\end{vmatrix}$
g $\begin{vmatrix}1&0\\0&-1\end{vmatrix}$	g $\begin{vmatrix}1&0\\0&-1\end{vmatrix}$	h $\begin{vmatrix}-1&0\\0&-1\end{vmatrix}$	e $\begin{vmatrix}1&0\\0&1\end{vmatrix}$	f $\begin{vmatrix}-1&0\\0&1\end{vmatrix}$
h $\begin{vmatrix}-1&0\\0&-1\end{vmatrix}$	h $\begin{vmatrix}-1&0\\0&-1\end{vmatrix}$	g $\begin{vmatrix}1&0\\0&-1\end{vmatrix}$	f $\begin{vmatrix}-1&0\\0&1\end{vmatrix}$	e $\begin{vmatrix}1&0\\0&1\end{vmatrix}$

$e \leftrightarrow e$
$a \leftrightarrow$
$b \leftrightarrow$
$c \leftrightarrow$

$b \times c = a$ ✓
$c \times b = a$
$a \times b = c$ ✓
$b \times a = c$

$g \times h = f$ ✓
$g \times f = h$ ✓

$g \leftrightarrow b$
$f \leftrightarrow a$
$h \leftrightarrow c$
} verificatis

Via inspection

$\begin{vmatrix}-1&0\\0&1\end{vmatrix}\begin{vmatrix}-1&0\\0&1\end{vmatrix} = \begin{vmatrix}1&0\\0&1\end{vmatrix}$

$f: G \leftrightarrow H$

$\begin{vmatrix}-1&0\\0&1\end{vmatrix}\begin{vmatrix}1&0\\0&-1\end{vmatrix} = \begin{vmatrix}-1&0\\0&-1\end{vmatrix}$ ✓ $\begin{vmatrix}1&0\\0&1\end{vmatrix} \leftrightarrow \begin{vmatrix}1&0\\0&1\end{vmatrix}$

$\begin{vmatrix}-1&0\\0&1\end{vmatrix}\begin{vmatrix}-1&0\\0&-1\end{vmatrix} = \begin{vmatrix}1&0\\0&-1\end{vmatrix}$ ✓ $\begin{vmatrix}-1&0\\0&-1\end{vmatrix} \leftrightarrow \begin{vmatrix}-1&0\\0&-1\end{vmatrix}$

$\begin{vmatrix}1&0\\0&-1\end{vmatrix}\begin{vmatrix}-1&0\\0&-1\end{vmatrix} = \begin{vmatrix}-1&0\\0&1\end{vmatrix}$ ✓ $\begin{vmatrix}0&1\\1&0\end{vmatrix} \leftrightarrow \begin{vmatrix}-1&0\\0&1\end{vmatrix}$

✓ $\begin{vmatrix}0&-1\\-1&0\end{vmatrix} \leftrightarrow \begin{vmatrix}1&0\\0&-1\end{vmatrix}$

Must do this properly

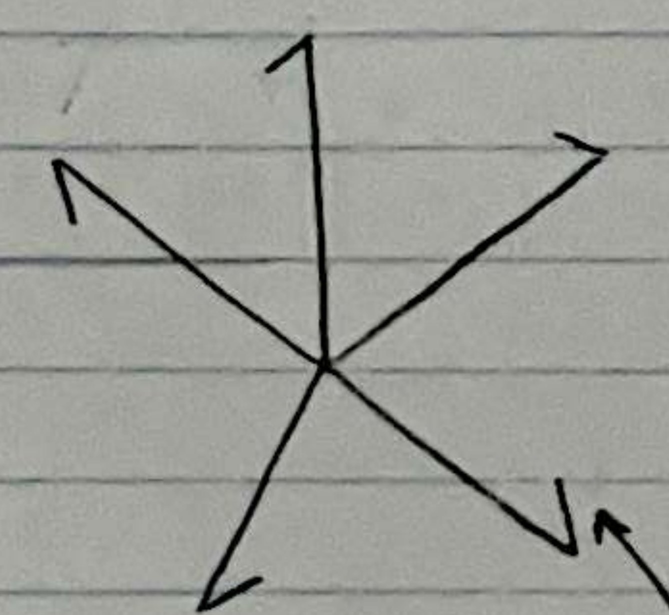
Q: ~~How~~ Is it important to know which elements map to which when studying group isomorphisms? How do we deduce this?

(f) $(\mathbb{Z}_5, +_5) \cong$ Symmetry group of 5-bladed windmill.

→ only $\frac{360^\circ}{5} = 72^\circ$ rotations

How to make a 5-bladed windmill using compass & straightedge?

→ YES. Measure angles & draw line.



No reflectional symmetry because of these.

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$e = id$

$r = 72^\circ$ cc/w

$s = 144^\circ$ cc/w

$t = 216^\circ$ cc/w

$u = 288^\circ$ cc/w

$\circ$	e	r	s	t	u
e	e	r	s	t	u
r	r	s	t	u	e
s	s	t	u	e	r
t	t	u	e	r	s
u	u	e	r	s	t

Via Inspection

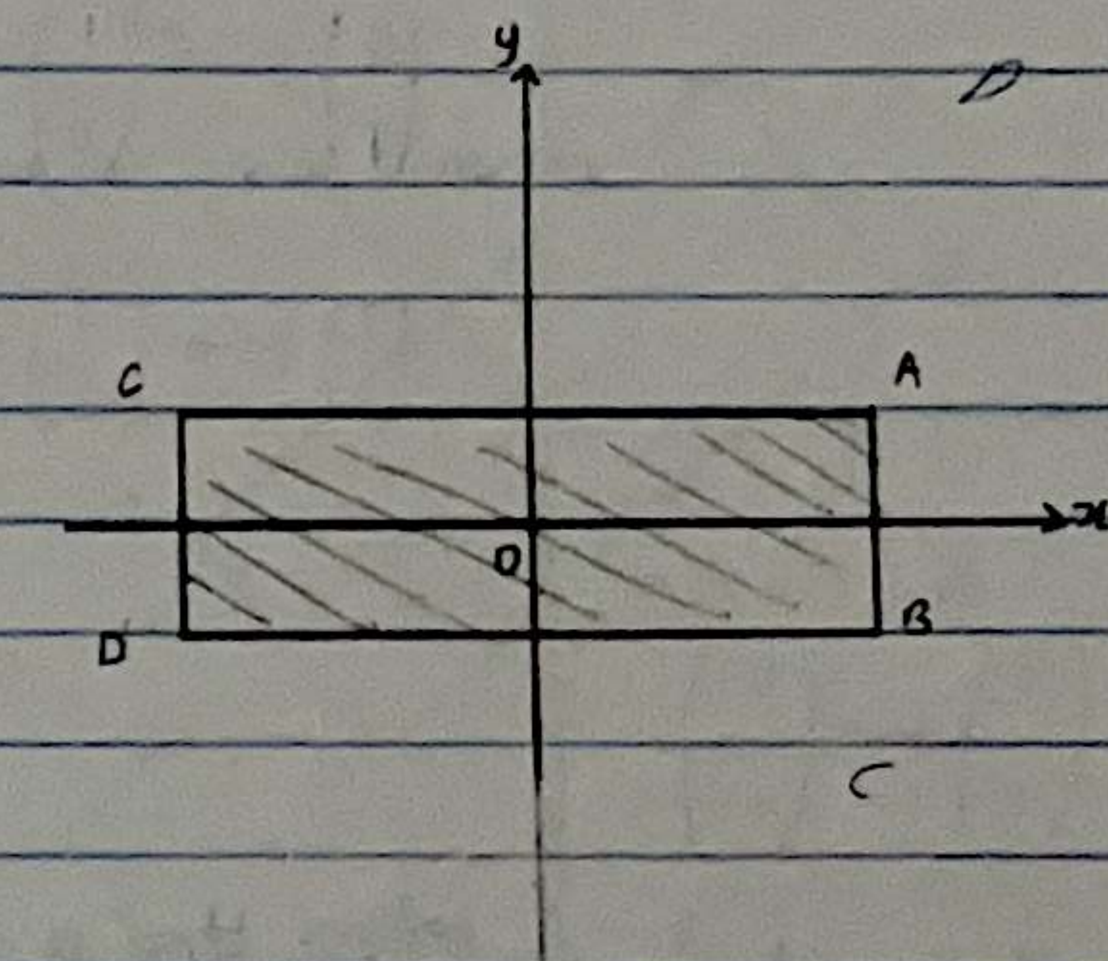
~~e ↔ 0~~ $0 \leftrightarrow e$ $\quad 4 \leftrightarrow u$

$1 \leftrightarrow r$

$2 \leftrightarrow s$

$3 \leftrightarrow t$

2 By positioning the rectangle in a suitable posn in the plane, establish that the Symmetric group of the rectangle is isomorphic to the Matrix groups in the previous exercise (1e).



Required?

(a) 1) The rectangle is placed in the plane with its geometric centre at the origin.

2) We then define the following (rigid) transformations according to the symmetry group of the rectangle. We label the corners ABCD as shown

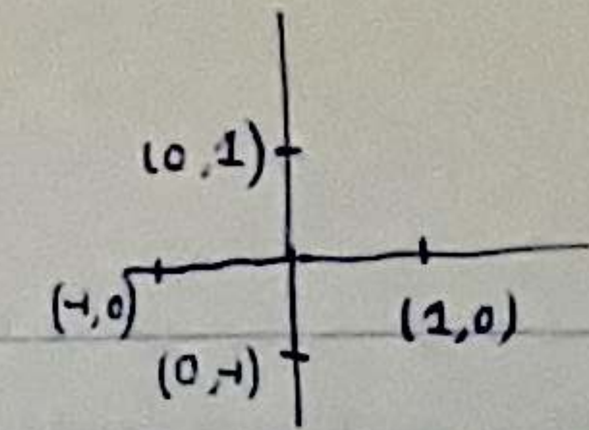
$e = id$

G $\quad x$ = reflect thru' x axis Transposes (A,B) AND (C,D)

y = reflect thru y-axis. Transposes (A,C) AND (B,D)

$S = 180^\circ$ cc/w rotation. Sends $A \to D$ $B \to C$ and vice versa.

3) We now study the effects of the matrix transformations on the plane



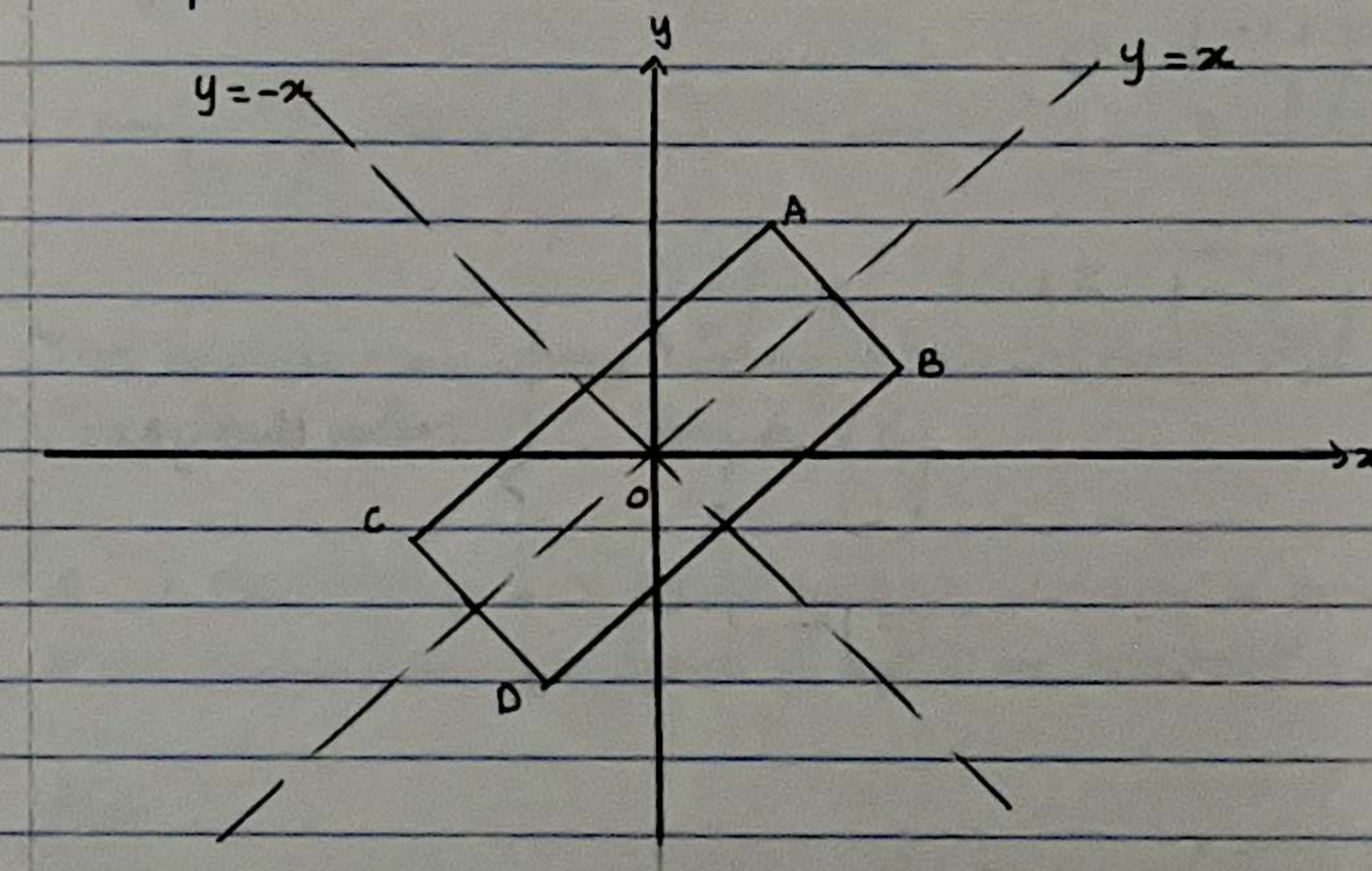
For ~~G~~ H

① $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$: id ② $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ s.t $\left.\begin{array}{l} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array}\right\}$ reflect thru' ~~x=y~~ $y=x$

③ $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ s.t $\left.\begin{array}{l} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ -1 \end{pmatrix} \end{array}\right\}$ reflect thru' $y=-x$

④ $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\left.\begin{array}{l} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ -1 \end{pmatrix} \end{array}\right\}$ 180° rotation.

We notice that we need to bring a correction to the orientation of the rectangle in the plane.



We now notice that applying a transformation to the plane with the rectangle oriented as shown has same effect as applying rigid transformations only to the rectangle.

ie. both e and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ do nothing (id)

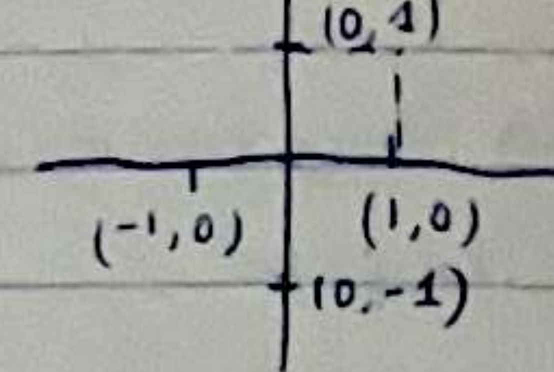
both x and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ transpose (A,B) AND (C,D)

both y and $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ transpose (A,C) AND (B,D)

both s and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ transpose (A,D) AND (C,B)

We can therefore define a map $f : G \to H$
f is one-to-one and onto (bijective)
and establish a correspondence between G and H
$\Rightarrow G \cong H$

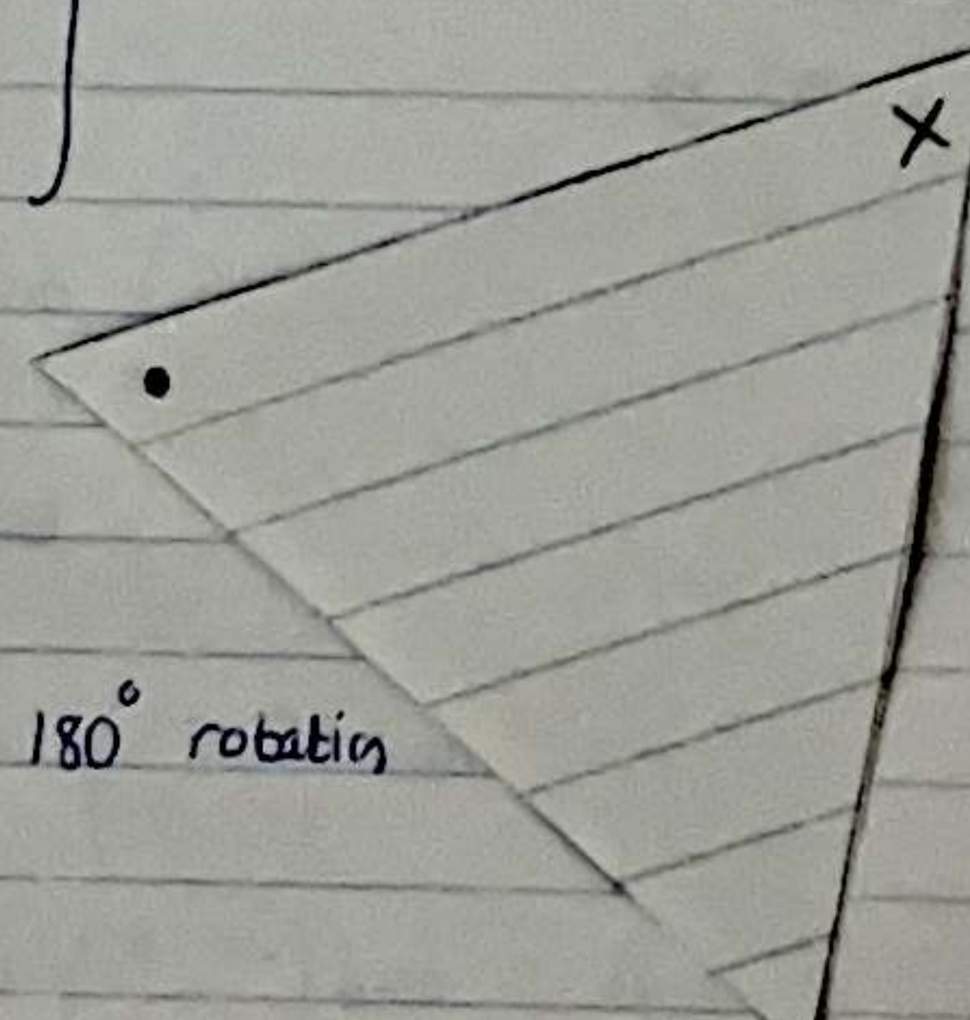
② 2^{nd} matrix group.



$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$: id

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ s.t $\left.\begin{aligned} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &\mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} &\mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} &\mapsto \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{aligned}\right\}$ reflect thru' y axis

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ s.t $\left.\begin{aligned} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &\mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} &\mapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} &\mapsto \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}\right\}$ reflect thru' x-axis



$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ s.t $\left.\begin{aligned} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &\mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} &\mapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} &\mapsto \begin{pmatrix} -1 \\ -1 \end{pmatrix} \end{aligned}\right\}$ 180° rotation

We can see that as we position our rectangle as we originally did, we get

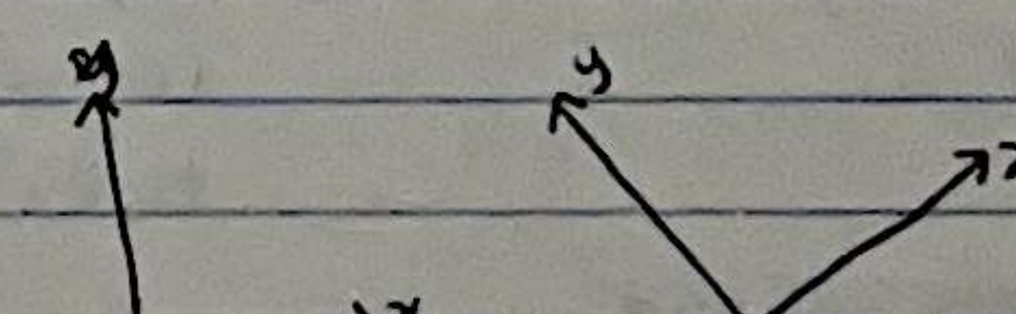
$e \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $x \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $y \leftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $s \leftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

The example above is very interesting as that case shows for 3 groups A, B and C if $A \cong B$ and $B \cong C$ then $A \cong C$

ie. if matrix group 1 ~~is~~ and 2 are both isomorphic to the symmetry group of the rectangle then matrix group 1 and 2 are isomorphic.

Also

We can see for 2 orientations of planes



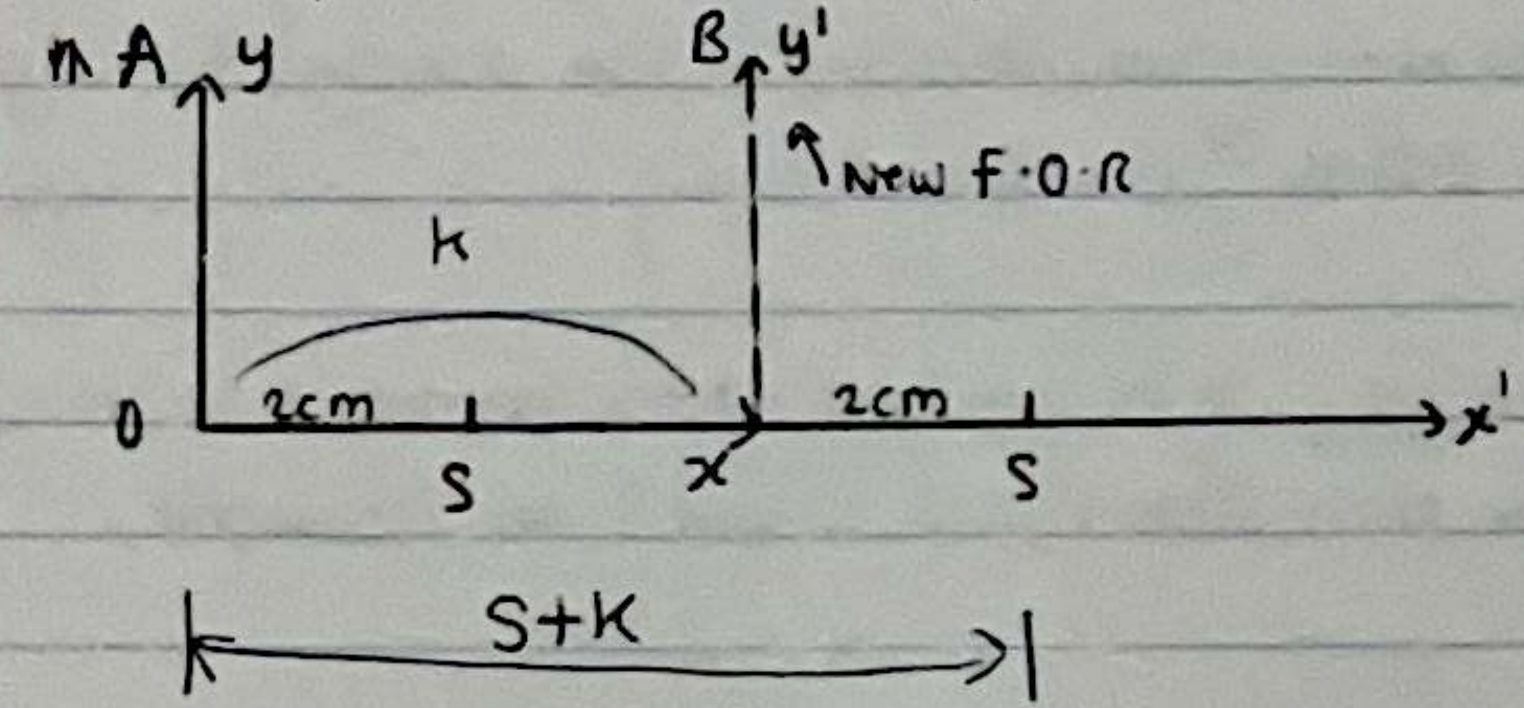
Group operations on them (Group action??) are isomorphic.
ie. All frame of references are isomorphic.
∴ Group invariants can be used to maybe determine invariants in for example Gallilean transformation ($F = ma$ 2nd law) and Lorentz transformations (c).

Also, Algebraic operations behave similarly in isomorphic groups.
This is why we had no trouble understanding that the reflection of the

Get SR from group Theory?

Original rectangle in the x-axis was similar to the reflection in $y=x$ in the diagonally oriented rectangle.

Also, Since algebraic operations are same across isomorphic groups, the equations of motions (eg. $v=u+at$) behave similarly across frame of reference provided the isomorphisms take the change into account.



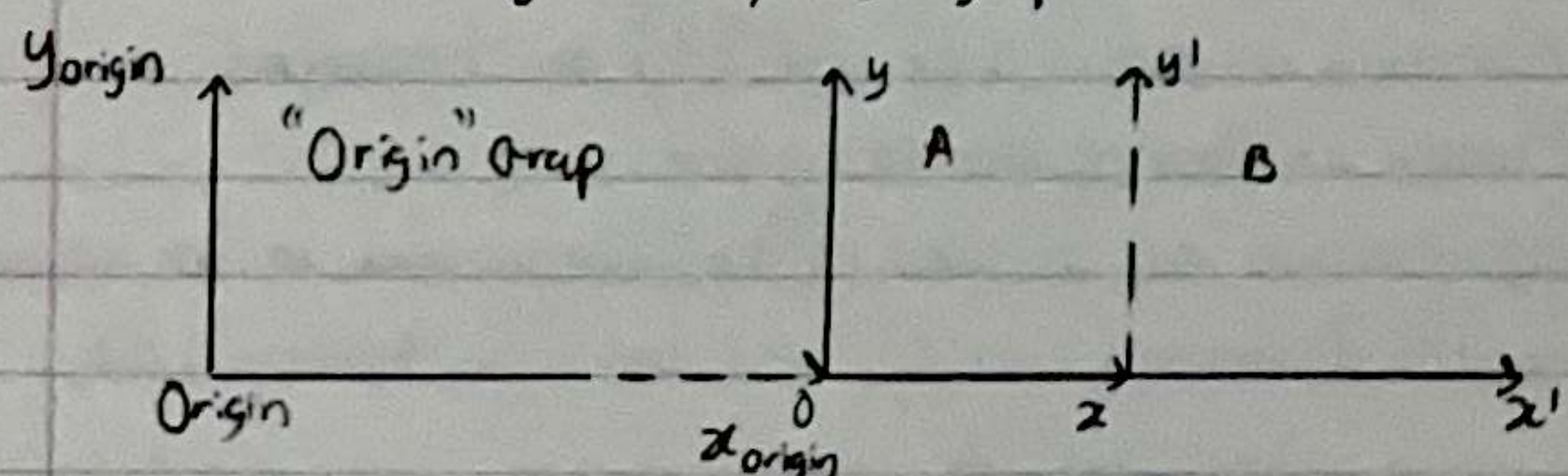
Mechanics problems can be worked out in either A or B and converted to the required f.o.r (Group) according to the isomorphism.

since $v=u+at$ is same across...

The isomorphism $f: A \to B$ takes S in A and maps it to (S+K) which is S in B but understood as (S+K) in A

In Category Theory??

But does a "higher" base/origin group exist



which is isomorphic to all groups similar to A & B and others can be generated from it?

3) By positioning the 3-bladed windmill suitably show that its symmetry group is isomorphic to the Isometry group $(\{e,r,s\},\circ)$ (14f (1))

G H

For $G(\{e,r,s\},\circ)$

Where $e = id$

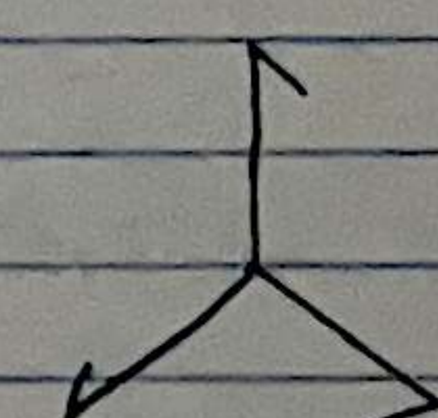
$r = 120^\circ$ cc/w rotation

$s = 240^\circ$ cc/w rotation

$\circ$	e	r	s
e	e	r	s
r	r	s	e
s	s	e	r

By geometric argument

1)



① The 3-bladed windmill is positioned with its centre at the origin in the plane.

Note: The figure has only rotational symmetries.

② We notice that as we transform the plane according to $e = id$, $r = 120^\circ$ cc/w, $s = 240^\circ$ cc/w the symmetries of the figure are realised.

$\Rightarrow$ The composition of all possible isometries of the plane correspond to the composition of all the possible symmetries of the figure

$\Rightarrow G \cong H$

4) (i) Prove that $\underbrace{\left(\left\{\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \,\middle|\, r \in \mathbb{R}^+\right\}, \times_m\right)}_{G} \cong \underbrace{(\mathbb{R}^+, \times_A)}_{H}$

$\times_m$: matrix multiplication

$\times_A$: Arithmetic multiplication.

We need to show that for an isomorphism f : $f(x \times_m y) = \cancel{f(x) *_A}$ $f(x) \times_A f(y)$ the latter holds.

$\forall a, b \in \mathbb{R}^+ : (a *_A b) \in \mathbb{R}^+$

Well known property of the reals.

Now

$$\underbrace{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}}_{f(a)} \times_m \underbrace{\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}}_{f(b)} = \underbrace{\begin{pmatrix} a \times_A b & 0 \\ 0 & a \times_A b \end{pmatrix}}_{f(a \times_A b)}$$

Arithmetic multiplication.

Since $f(a \times_A b) = f(a) \times_m f(b)$

$\Rightarrow$ Better proof.

~~f(a)~~ =

For any $r \in \mathbb{R}^+$

$\exists f \mid f : G \leftrightarrow H$

$f(r) = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$

f sends any real number r to $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$

$\forall a, b \in \mathbb{R}^+ \mid (a \times_A b) \in \mathbb{R}^+$

$f(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $\quad f(a \times_A b) = \begin{pmatrix} a \times_A b & 0 \\ 0 & a \times_A b \end{pmatrix}$

$f(b) = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$

$f(a) \times_m f(b) = f\begin{pmatrix} a \times_A b & 0 \\ 0 & a \times_A b \end{pmatrix}$

$\downarrow$

$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \times_m \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$

$\Rightarrow G \cong H$

$$\underbrace{\left(\left\{\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \mid r \in \mathbb{R}^+\right\}, \times_m\right)}_{H} \cong \underbrace{(\mathbb{R}^+, \times_A)}_{G}$$

(ii) Prove that $\left(\left\{\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \mid r \in \mathbb{R}^+\right\}, \times_m\right) \cong (\mathbb{R}^+, \times_A)$

$\forall a, b \in \mathbb{R}^+ \mid (a \times_A b) \in \mathbb{R}^+$ f sends the reals to the Matrix form.

We define an isomorphism $f : G \rightarrow H$ Where f is one-to-one and onto.
Such that ~~$f(a \times_m b) = f(a) \times_A$~~ (bijective).

$$\boxed{f(a) \times_m f(b) = f(a \times_A b)}$$

$f(a) = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ $\quad f(b) = \begin{pmatrix} b & 0 \\ 0 & 1/b \end{pmatrix}$

$f(a \times_A b) = \begin{pmatrix} a \times_A b & 0 \\ 0 & 1/a \times_A b \end{pmatrix}$

When we evaluate

$f(a) \times_m f(b) = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \times_m \begin{pmatrix} b & 0 \\ 0 & 1/b \end{pmatrix} = \begin{pmatrix} a \times_A b & 0 \\ 0 & \frac{1}{a \times_A b} \end{pmatrix}$

Which equals
$f(a \times_A b)$

Since $f(a) \times_m f(b) = f(a \times_A b)$

$\Rightarrow G \cong H.$

Define with better & absolute explanation

Now lets work for the matrix.

ie. $f : H \rightarrow G$
that is f sends r in the matrix $\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$ to r on the real line.

~~$f\left(\begin{vmatrix} a & 0 \\ 0 & 1/a \end{vmatrix} \times_m \begin{vmatrix} b & 0 \\ 0 & 1/b \end{vmatrix}\right) = f(a) \times_A f(b)$~~ $\quad f(a \times_A b) = f\left(\begin{vmatrix} a \times_A b & 0 \\ 0 & \frac{1}{a \times_A b} \end{vmatrix}\right)$

$f\left(\begin{vmatrix} a & 0 \\ 0 & 1/a \end{vmatrix}\right) \times_m f\left(\begin{vmatrix} b & 0 \\ 0 & 1/b \end{vmatrix}\right) = f(a \times_A b)$

Action of f as defined

$f(a) \quad \times_m \quad f(b)$

$\Rightarrow f : H \leftrightarrow G$
(bijective)

5) Use the function e^x and the exponentiation law $e^x e^y = e^{x+y}$ to prove that $(\mathbb{R}^+, +) \cong (\mathbb{R}^+, \times)$

We define a function $f : (\mathbb{R}^+, \times) \to (\mathbb{R}^+, +)$
Where $f(x) = e^x$

For f to be an isomorphism $\cancel{f(a \times b)} = \cancel{f(a+b)}$ needs to hold.

$\cancel{f(a)+f(b)}$

$f(a) \times f(b) = f(a+b)$

$f(x) = e^x$
$f(y) = e^y$
$f(x+y) = e^{x+y}$

$\cancel{f(x) \times f(y) = f}$

We know
$e^x \times e^y = e^{x+y}$ (exponentiation law) $\longrightarrow$ Reverse this eq$\underline{n}$ to work backwards.

$\Rightarrow f(x) \times f(y) = f(x+y)$

$\Rightarrow (\mathbb{R}^+, \times) \cong (\mathbb{R}^+, +)$

Now in reverse

$f : (\mathbb{R}^+, +) \to (\mathbb{R}^+, \times)$

$f(a) + f(b) = f(a \times b)$ Work on this

$\downarrow$

$\cancel{f: e^{x+y} \to e^x \times e^y}$

The inverse is taking logs.

f sends the reals under addition to the reals under multiplication.

We define $g : (\mathbb{R}^+, +) \to (\mathbb{R}^+, \times)$

and $f : (\mathbb{R}^+, \times) \to (\mathbb{R}^+, +)$

And $f = g^{-1}$

If $f = e^x$
then $g = \ln x$

$$\left.\begin{aligned} f \circ g &= e^{\ln x} = x \\ g \circ f &= \ln e^x = x \end{aligned}\right\} \text{Involution}$$

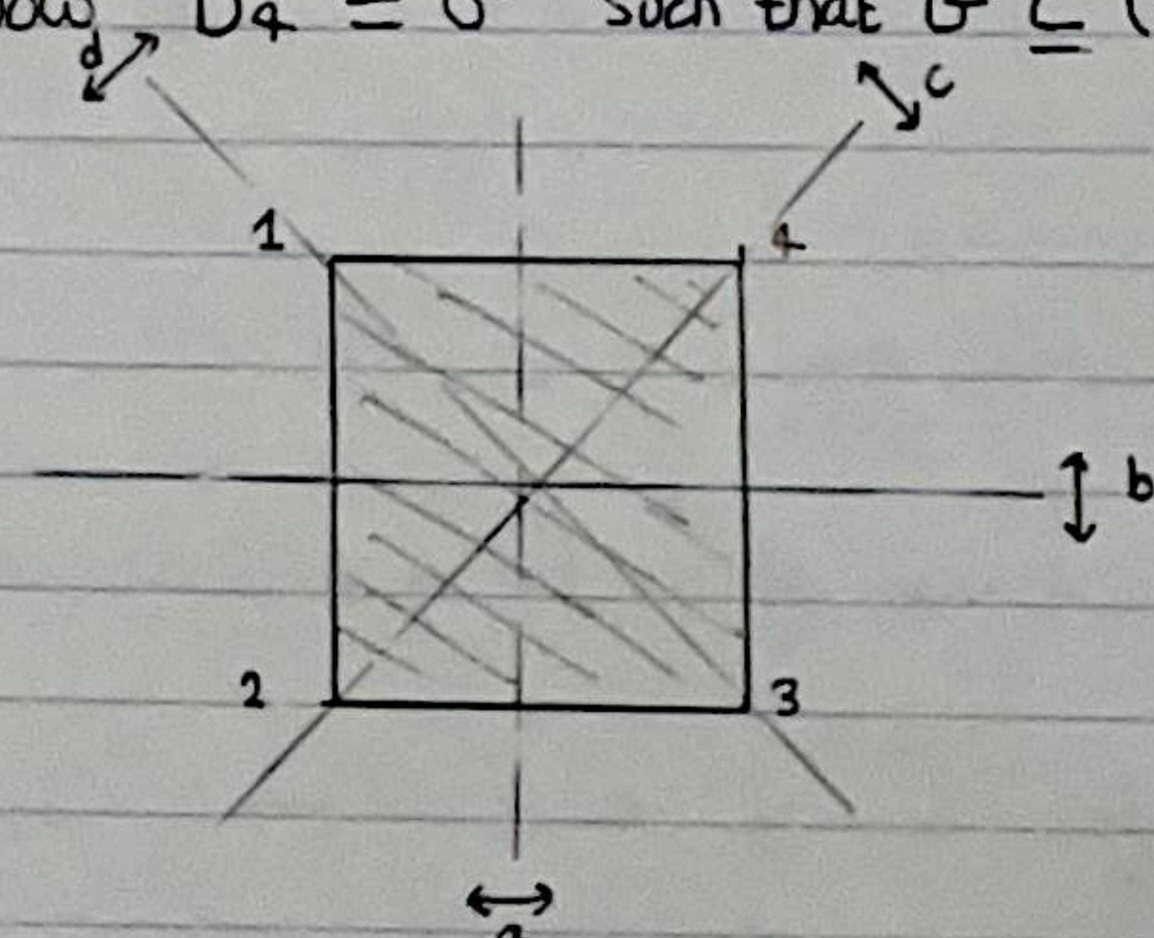
Dihedral group degree 4 /order 8.

6) By labelling the corners of the <u>square</u> as shown, write down a set of <u>eight</u> elements of S_4 which form a subgroup of $(S_4, \circ)$

Permutation group on 4 Symbols.

$|S_4| = 4! = 24$

We need to show $D_4 \cong G$ such that $G \subseteq (S_4, \circ)$



The possible isometries of D_4 are e, r, s, t, a, b, c, d

Where $e = id$, $r =$ cc/w 90°, $s = 180^\circ$ cc/w, $t = 270^\circ$ cc/w, $a =$ ref. thru a

$b =$ " " b

$c =$ " " c

$d =$ " " d

We notice that

e does nothing to the corners ie. it is the identity permutation.

a sends 2 to 3 and 1 to 4 and vice versa

and so on...

The composition of isometries correspond to the composition of permutations.

Note: G acts on the corners of the labelled square and not the sides.

$G: e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ $\quad r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ $\quad s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

$b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ $\quad t = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

$c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

$d = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$ $\quad r \circ s \doteq t = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$ ✓✓

Check

7) Check and prove that all groups with 2 elements in them are isomorphic.

$G(\{e_G, a\}, \circ)$ $\qquad$ $H(\{e_H, b\}, *)$ $\qquad$ form or only for function.

$a = a^{-1}$ $\qquad$ $b = b^{-1}$

like reflections ← $a \circ a^{-1} = a^{-1} \circ a = e$ $\qquad$ $b \circ b^{-1} = b^{-1} \circ b = e$ (Involution?!)

$\circ$	e_G	a
e	e_G	a
a	a	e_G

$*$	e_H	b
e	e_H	b
b	b	e_H

Should the equation for defining isomorphism be in the form $h(u \circ v) = h(u) * h(v)$ or $h(v) \circ h(u)$

Study this better

Let G and H be any 2 arbitry groups of order 2.

Let f be an isomorphism $f: G \to H$

We know isomorphisms sends the identity of one group to another.

ie. $f(e_G) = e_H$

Then the only remaining possibility is that $f(a) = b$

~~$f(e_G \circ a) = f(e_H) * f(b)$~~ must be same element of 1st group.

~~$f(a) = e_G * f(b)$~~

~~$f(e_G) \circ f(a) = f(e_H * b)$~~

Rework previous exercises stuck

$f(e_G \circ a) = f(e_G) * f(a)$

$f(a) = e_H * f(a)$

The only remaining element is b

$\Rightarrow f(a) = b$

For isomorphism to hold for $G(\{u, v\}, \circ) \cong H(\{a, b\}, *)$ → in H.

$f(u \circ v) = f(u) * f(v)$

In the previous case

$f(e_G \circ a) = f(e_G) * f(a)$

We know $f(a) = b$ since b is the only possibility remaining.

$f(a) = e_H * f(a)$

$b = e_H * b$

$= b$

$\Rightarrow G \cong H$.

Now same process is applied if $H \cong M$ where M is another group of order 2.

we know If $A \cong B$ and $B \cong C$ then $A \cong C$ → More rigorous proof or is this good enough?!

~~By infinite (descent)~~

$\Rightarrow G \cong H \cong M (\ldots)$ to infinity

8) Check by using tables that the ~~Sy~~ Subgroup $(\{e,r,s,t\},\circ)$ of the Symmetry group of the square is isomorphic to $(\mathbb{Z}_4, +_4)$ and by using Q6 write down a subgroup of $(S_4, \circ)$ which is also isomorphic to $(\mathbb{Z}_4, +_4)$ and check using tables.

$G(\{e,r,s,t\},\circ)(\subseteq D_4)$

$\circ$	e	r	s	t
e	e	r	s	t
r	r	s	t	e
s	s	t	e	r
t	t	e	r	s

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Via inspection

$\exists f \mid f$ is an isomorphism $f: G \to (\mathbb{Z}_4, +_4)$

$f:$

$\left.\begin{array}{l} e \to 0 \\ r \to 1 \\ s \to 2 \\ t \to 3 \end{array}\right\}$

Now for $(S_4,\circ)$ we need a subgroup of the permutation group on 4 symbols.

We know If $A \cong B$ and $B \cong C$ then $A \cong C$

From Q6

$e = 0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = id$ $\quad t = 3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = (1,4,3,2)$

$r = 1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \cancel{(1 \to 2}$ $(1,2,3,4)$

$s = 2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (1,3)(2,4)$

Table

$\circ$	id	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)
id	id	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)
(1,2,3,4)	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)	Id
(1,3)(2,4)	(1,3)(2,4)	(1,4,3,2)	id	(1,2,3,4)
(1,4,3,2)	(1,4,3,2)	id	(1,2,3,4)	(1,3)(2,4)

$(1,2,3,4) \circ (1,2,3,4)$

$= (1,3)(2,4)$ ✓✓

Algebraic Manipulations in Groups

NOTE: For ease of representation group operations are sometimes omitted.

Note: by convention $a^0 = e$

$a^2 \Rightarrow$ any element of a group composed with itself twice

ie. $\forall a \in G \mid G(\{e, a, \ldots\}, \circ) \Rightarrow a^3 = a \circ a \circ a$

for additive groups

eg $(\mathbb{Z}, +)$

$\forall a \in (\mathbb{Z}, +)$ $a^3 \Rightarrow a + a + a$

Group operation (NOT exponentiation).

$a^{-1} = -a$

Solution of equations in groups.

1) Solve x for the equation

$ax = b$

1) multiply (compose left) both sides by a^{-1}

$\Rightarrow a^{-1}ax = a^{-1}b$

$ex = a^{-1}b$ (since $a^{-1}a = e$) $\rightarrow e \circ x = a^{-1} \circ b$

$x = a^{-1}b$ (since $ex = x$)

2) Show that the solution $x = a^{-1}b$ for $ax = b$ is unique

Suppose $ay = b$ and since $ax = b$

$\Rightarrow ay = ax$

Compose L.H.S with a^{-1} on both side of eqn

$a^{-1}ay = a^{-1}ax$

$\Rightarrow ey = ex$

$\Rightarrow y = x$

ie. any other solution is equal to x

$\Rightarrow x = a^{-1}b$ is unique.

Try this argument for 2^0 eqn.

Note: $ax = ay \Rightarrow x = y$

we could compose a^{-1} on LHS

for $xa = ya$ we compose a^{-1} on RHS

These are known as group cancellation laws.

Note: $a^{-1}a = aa^{-1} = e$

Composition of an element with its inverse is commutative.

also $ax = ya$ cannot be solved in this way if group is not Abelian.

3) Given that $ab = e$, prove that $a = b^{-1}$ and $b = a^{-1}$

$ab = e$	$ab = e$
Compose a^{-1} on LHS of both side of eqn	compose b^{-1} on RHS on both side of eqn.
$a^{-1}ab = a^{-1}e$	$abb^{-1} = eb^{-1}$
$eb = a^{-1}e$	$ae = eb^{-1}$
$b = a^{-1}$ ←	$a = b^{-1}$ ←

The above example explains why e occurs exactly once in any row g of a group table and that it occurs in the g^{-1} column. If e occurs in row g and column h then $gh = e$ and $g = h^{-1}$. In fact, each element of a group occurs exactly once in any row of a group table.

why??

4) Prove that each element of a group occurs exactly once in any given row of the group table.

$ax = b$ in words : element b occurs in row a column x.

but as we have seen above there ~~to~~ is an unique solution for x.

ie. for a given row a and a given element b there is an unique column x in which it occurs. ie. each element occurs exactly once in a row.

$\circ$	- - -	x		
⋮				
⋮				
a		b		

$ax = b$

- we have shown x is unique for a particular value of a & b
- Now in the group we list all the existing values of a & b in the cayley Table.

$\circ$	a	b	c	d
a		d	d	
b				
c				
d				

$\Rightarrow a \circ b = d$
$a \circ c = d$
$\Rightarrow ab = ac \Rightarrow b = c$

Contradiction as all elements of a set are unique.

$\Rightarrow$ ~~There will be~~ ie. if we fix a (ie. a considering a row) and we allow b to vary. The value x takes will depend solely on b.

Since b takes different values in different column x will also take different values.
- Now $x \in G$ (for closure)

$\Rightarrow$ All possibilities of a group occurs in a row exactly once.

In a row eg. for a no other element compose with a the same way else they would be identical

5) Given that $(ab)^{-1} = a^{-1}b^{-1}$ holds $\forall (a,b) \in G$ pair of elements (a,b), Prove that G is commutative (Abelian).

$(ab)^{-1} = a^{-1}b^{-1}$

Compose LHS of equation by (ab)

$(ab)(ab)^{-1} = (ab)a^{-1}b^{-1}$

$\Rightarrow (ab)a^{-1}b^{-1} = e$

$= aba^{-1}b^{-1} = e$ (Remember order must be respected and since associativity holds in the group $\Rightarrow$ Posn of bracket is immaterial {})

Compose RHS of equation with b

$aba^{-1}b^{-1}b = eb$

$aba^{-1} = b$

BUT : a and b must not be separated
ie. $(ab)(cd) = ((ab)c)d = (a(bc))d = abcd$.

Compose RHS of eqn with a

$aba^{-1}a = ba$

$\Rightarrow ab = ba$ Since this holds for all pair of elements a & b in g

$\Rightarrow$ Commutative.

Why associativity implies brackets don't matter but composition laws must ??

Associativity

~~ab~~

$(a \circ b) \circ c = a \circ (b \circ c)$

Commutative

$a \circ b = b \circ a$

For commutative, order of composition does not matter
ie, $a \circ b \Rightarrow$ b first then a
$b \circ a \Rightarrow$ a first then b
for abelian
$a \circ \underline{b} = b \circ \underline{a}$
$\Rightarrow$ Answer is same either if left or right is done first.

This is NOT the case for associative as the order of composition must be obeyed

For $(a \circ b) \circ c \Rightarrow$ c first then $(a \circ b)$.

$a \circ (b \circ c) \Rightarrow (b \circ c)$ first then a.

Associative $\Rightarrow$ The grand net result of a complex operation is an same for any combination of elements present provided order of composition is respected.

14 i

1) Check the following cases of the law of indices.

(a) $a^{-2}a^{3} = a$

~~LHS by a^2~~

$\cancel{a^2 \underbrace{a^{-2}}_{e} a^3 = a^2 a}$

$\cancel{a^3 = a^2 a}$

$\underbrace{a^{-1}a^{-1}}aaa \stackrel{!}{=} a$

$a^{-1}eaa = a$

$a^{-1}aa = a$

$ea = a$

$a = a \leftarrow$

(b) $a^{4}a^{-2} = a^{2}$

$aa\underbrace{aa^{-1}}_{e}a^{-1} \stackrel{!}{=} a^{2}$

$aaaea^{-1} = a^2$

$aa\underbrace{aa^{-1}}_{e} = a^2$

$aae = a^2$

$aa = a^2$

2) (a) ~~Simplify~~ $\cancel{a^2 e a^{-1} a^2 a^4}$

~~Compose a^{-2} LHS~~

$\cancel{a^{-2} a^2 e a^{-1} a^2 a^4} = \cancel{e e a^{-1} a^2 a^4}$

Simplify $a^2 e a^{-1} a^2 a^4$

Since $a^2 \circ e = a^2$

$= a^2 a^{-1} a^2 a^4$

→ or simplify to lowest power

??

Now $a^2 \circ a^{-1} = a$

$\Rightarrow a a^2 a^4$

$= a^3 a^4$

$= a^7 \leftarrow$

(b) Given $ab = e$

Simplify $a^2 b a^{-1} b e a^2$

$a a b a^{-1} b e a a$

$a a b a^{-1} b a a$

$a e a^{-1} b a a$

$a a^{-1} b a a$

$\underbrace{a a^{-1}}_{e}$

$b a a$

Now $ab = e$

$\Rightarrow a = b^{-1}$

$\Rightarrow ab = ba$ (composition of a with its inverse can be commutative)

$\underbrace{ba}_{e} a$

$= a$

3) Prove that the equation $ya = b$ has a unique solution for y in the grap G.

Compose RHS of eqn with a^{-1}

$y a a^{-1} = b a^{-1}$

$y e = b a^{-1}$

$y = b a^{-1}$

~~Let $y = x$~~ Suppose $xa = b$

$\Rightarrow xa = b$

$\Rightarrow xa = ya$

Compose RHS with a^{-1}

$x = y$

ie. Any other soln is equal to y

$\Rightarrow y = ba^{-1}$ is unique.

4) Prove that by the same method as example 3

~~$(a^{-1})^{-1} = a$~~

~~Multiply right side by (a^{-1})~~

~~$(a^{-1})^{-1}(a^{-1}) = a(a^{-1})$~~

~~$e = e$~~

$(a^{-1})^{-1} = a$

multiply on right side by (a^{-1})

$(a^{-1})^{-1}(a^{-1}) = a(a^{-1})$

$e = e$

~~Prove $(a^{-1})^{-1} = a$~~

$(a^{-1})^{-1}(a^{-1}) = \underbrace{a a^{-1}}_{e}$

Prove $(a^{-1})^{-1} = a$

Multiply $(a^{-1})^{-1}$ on right by (a^{-1})

$\Rightarrow (a^{-1})^{-1}(a^{-1}) = e$

now multiply RHS by a

$(a^{-1})^{-1}\underbrace{(a^{-1})a}_{e} = ea$

$\Rightarrow (a^{-1})^{-1} = a \leftarrow$

Q: Prove $(a^{-1})^{-1} = a$

In these cases of question we first consider only the LHS Portion

$(a^{-1})^{-1} \leftarrow$ Expression

We compose the expression with an element of our choice and show that result after an equal sign (NOT the first eqn!!)

$(a^{-1})^{-1}(a^{-1}) = e \leftarrow$ "Auxilliary eqn"

We then do futher operations to show how the "Auxiliary equation" becomes the first eqn.

$(a^{-1})^{-1}\cancel{(a^{-1})a} = ea$

$(a^{-1})^{-1} = a \leftarrow$ First eqn.

Prove $(ab)^{-1} = b^{-1}a^{-1}$

Compose $(ab)^{-1}$ with (ab) on RHS.

$(ab)^{-1}(ab) = e$

Compose $(ab)^{-1}(ab)$ with b^{-1} on RHS $\rightarrow$ Show Commutativity.

$(ab)^{-1}(ab)b^{-1} = eb^{-1}$

Compose with a^{-1}

$(ab)^{-1} = eb^{-1}a^{-1}$

$(ab)^{-1} = b^{-1}a^{-1}$

5) Given either that $fa = a$ or $af = a$ prove $f = e$. Deduce the identity in a group is unique

~~$fa(a^{-1}) = fe$~~

Compose RHS with a^{-1}

~~$faa^{-1} = fe$~~

$fa = a$

$faa^{-1} = aa^{-1}$

$fe = e$

$f = e$

~~Suppose $fa = a$~~

~~$faa^{-1} = a^{-1}$~~

~~$f = a^{-1}$~~

Not like previous example.

In that case we do not prove an equation but have to work with the equation given

~~Suppose group operation is row ∘ column~~

$fa = a$

$faa^{-1} = aa^{-1} = e$

$\Rightarrow f = e$

Suppose ~~f~~ $ga = a$

$fa = ga$

Group cancellation law

$f = g$

$\Rightarrow$ All values of the soln is equal to f

$\Rightarrow f$ is unique.

Now $af = a$

$a^{-1}af = a^{-1}a$

$f = e$

Suppose $ag = a$

$ag = af$

$g = f$

$\Rightarrow f$ is unique

$\Rightarrow$ ~~Eq~~ It can be interpreted as if group operation is row ∘ column $\Rightarrow f = e$

and column ∘ row $\Rightarrow f = e$

$\circ$	f
f	f

only the identity has this property in the table.

6 Prove that each element in a group occurs only once in each column of the group table.

~~ie. prove~~ ~~$ax = b$~~

$xa = b$

~~Where x is unique and group operation is (Column $\circ$ row)~~

~~$x = ba^{-1}$~~

~~↳ prove that element b occurs only once in column x.~~

~~for a given column x and given element b~~

$ax = b$

(row $\circ$ Column)

$\underline{\underline{x = a^{-1}b}}$

x is unique

ie.

for a given column x and given element b there is an unique row a in which b occurs.

-As we fix x and allow a to vary the ~~uniq~~ value of b is unique to each row a

⇒ All elements occur once in a column.

→ Unique but does it imply ALL elements ??

↳ If there are n number of rows

⇒ there will be n number of elements in a column.

Since all elements are unique there will be n unique elements.

Suppose $n = 3$

Table is just comparing different ways of elements of set with themselves.

$\circ$	a	b	c
a			
b			
c			

All elements of set {a, b, c}

Column

Other way ??

We proved in the previous example that all elements in a row are unique.

G

$\circ$	a	b	c
a	–	–	–
b	–	–	–
c	–	–	–

In the g^{th} row and h^{th} column of the group table we find gh

φ ~~sends rows~~ exchanges rows and columns

Now let φ be an automorphism of G. φ is applied to the labels and entries of the Cayley table.

Now in $\varphi(g)^{th}$ row and $\varphi(h)^{th}$ Column we have $\varphi(gh)$

Now $\varphi(g) = h$

$\varphi(h) = g$

$\varphi(gh) = hg$

Now $\varphi(g)\varphi(h) = \varphi(gh)$ → Homomorphism (φ)

Apart from the permutation (relabelling rows & columns) We get the original Cayley Table back.

7) Use the fact that each element in a group occurs exactly once in each row and column to complete the group table.

$\circ$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

→ How did I do it (Precisely)?

Are all groups like these abelian?

Deduce that any 2 group with 3 elements in them are isomorphic.

$(G, \circ)$

$\circ$	e_G	a	b
e_G	e_G	a	b
a	a	b	e_G
b	b	e_G	a

$(H, *)$

$*$	e_H	c	d
e_H	e_H	c	d
c	c	d	e_H
d	d	e_H	c

We say let $ab = e$

and $aa = b$

and $bb = a$ NOTE !!

what if $aa = \underline{\underline{a}}$

↑ Impossible as it is the property of the identity

Same for b.

Let $f : G \to H$ where f is bijective.

$f(a \circ b) = f(a) * f(b)$

$f(e_G) = c * d$

$e_H = e_H$ ✓✓

$\Rightarrow G \cong H$.

(Also, their Cayley table is the same)

for Correspondence $\left.\begin{array}{l} e_G \to e_H \\ a \to c \\ b \to d \end{array}\right\}$

Check if f is one-to-one.

Injective : If $f(x) = f(y) \Rightarrow x = y$

↑ domain ↑ codomain

$f(a) = f(c) \Rightarrow a = c$ (True)

$\Rightarrow$ One-to-one

Surjective

$f : G \to H$

$\forall z \in H \quad z = f(k) \mid k \in G$ → Better explanation.

$k = f^{-1}(z) \Rightarrow k \in G \to$ True

↳ Returns to group G

$\Rightarrow$ onto

$\Rightarrow$ f is ~~bb~~ bijective

How do we know that it is truly correct?

$ac = e$

$bb = e \Rightarrow b = b^{-1}$

$ca = e \quad a = c^{-1}$

$c = a^{-1}$

8) Complete the group tables

(a) $(G, \circ)$

$\circ$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

(b) $(H, *)$

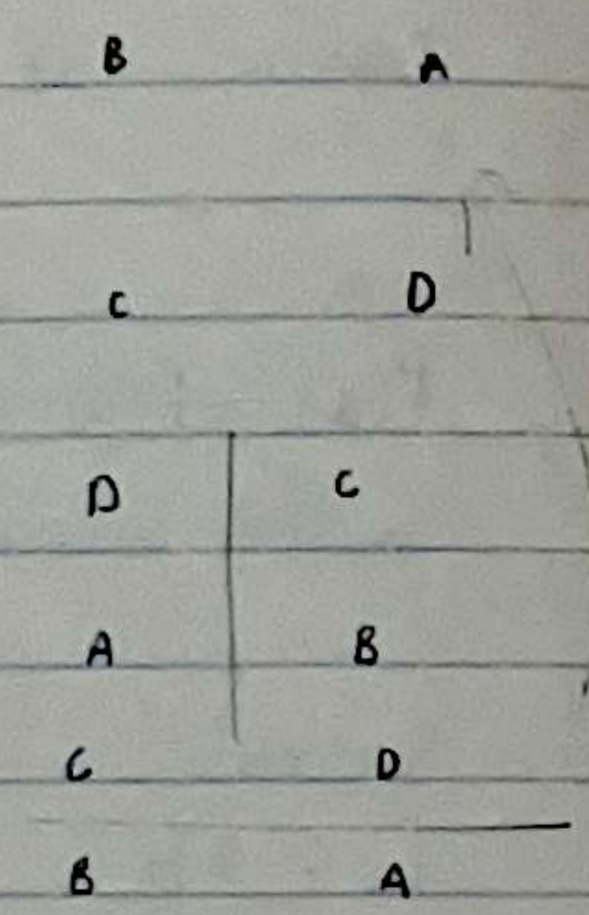
$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Let $aa = b$

$ab = c$

$bc = a$

Find out which of these two previous group is isomorphic to $(\mathbb{Z}_4, +_4)$ or $S(R)$ → Symmetry group of the rectangle.



$(\mathbb{Z}_4, +_4)$

↓

(b)

$$\left.\begin{array}{l} e \to 0 \\ a \to 1 \\ b \to 2 \\ c \to 3 \end{array}\right\}$$

$\Rightarrow$ ~~$(G, \circ)$~~ $(H, *) \cong \mathbb{Z}/_{4\mathbb{Z}}$

$S(R)$

↓

(a)

$$\left.\begin{array}{l} e \to id \\ a \to \text{ref. in } x\text{-axis} \\ b \to \text{ref. in } y\text{-axis} \\ c \to 180^\circ \text{ rotation} \end{array}\right\}$$

↘ $(G, \circ) \cong S(R)$

↘ Symmetry Group

S_n → Permutation Group

Notation : $\mathbb{Z}/_{4\mathbb{Z}}$

- It is the same as integer-modulo 4
- read as "$\mathbb{Z}$ over $4\mathbb{Z}$" $\mathbb{Z}$ mod 4
- Can be called a quotient group.
  → A quotient group or factor group is a mathematical group obtained by aggregating similar elements of a large group using an equivalence relation that preserves some of the group structure. (the rest of the structure is "factored" out.)

↘ Study Equivalence classes

↗ Group of "remainders" modulo n

$\mathbb{Z}/_{n\mathbb{Z}}$

- Consider the group of integers $\mathbb{Z}$
- Consider the sub-group $n\mathbb{Z}$ (multiples of n)

$\mathbb{Z} = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16$

$4\mathbb{Z} = 4, 8, 12, 16, \ldots$

Quotient → Quot - Latin for how many ↗ Also : degree/amount of a specified quantity/characteristic : math division.

$8/_2 = 4$ $\quad \dfrac{\text{dividend}}{\text{divisor}} = \text{quotient}$

↘ can be read as "How many 2s are there in 8"

equivalence relation $\boxed{4m \pmod 4 = 0 \quad \forall m \in \mathbb{Z}}$??

↘ Correct notation ??

9) Given $a^2b^2 = (ab)^2$ hold for each pair of elements a, b in a group G. Prove that G is commutative.

$\cancel{a^2b^2(ab)^{-2} = (ab)^2(ab)^{-2}}$

$\cancel{aabb}$

$\cancel{a^2b^2 = (ab)^2}$

$\cancel{a^2b^2 =}$

$\cancel{a^2b^2(ab)^{-2} = (ab)^2(ab)^{-2}}$

$a^2b^2 = (ab)^2$

$aabb = (ab)^2$

$a^{-1}aabb^{-1} = a^{-1}(ab)^2b^{-1}$

$ab =$

$a^2b^2 = (ab)^2$

$aabb = (ab)(ab)$

$a^{-1}aabbb^{-1} = a^{-1}(ab)(ab)b^{-1}$

Associativity holds

ros ⇒ S first then r

$\cancel{a^{-1}(ab)ae}$

$a^2b^2 = (ab)^2$

$aabb = (ab)(ab)$

$\cancel{a^{-1}}aab\cancel{b}b^{-1} = a^{-1}(ab)(ab)b^{-1}$

$ab = \cancel{a^{-1}a}ba\cancel{bb^{-1}}$

$= ba$

Since $ab = ba$

⇒ G is Abelian ←

Redo Exercise

10) Given that $a^3b^3 = (ab)^3$ for each pair of element a, b in group G, prove that $a^2b^2 = (ab)^2$. Deduce (or prove otherwise) that $a^4b^4 = (ab)^4$ and $(ab)^3 = (ba)^3$

① $a^3b^3 = (ab)^3$

⇒ $aaabbb = (ab)(ab)(ab)$

$\cancel{a^{-1}a}aabb\cancel{bb^{-1}} = a^{-1}(ab)(ab)(ab)b^{-1}$

$eaabbe = \cancel{a^{-1}(ab)b^{-1}}(ab)(ab)$

$a^2b^2 = (ab)^2$ ← ①

for normal

$aabb = (ab)(ab)$

$a^{-1}abbb^{-1} = a^{-1}(ab)(ab)b^{-1}$

$= \cancel{a^{-1}a}ba\cancel{bb^{-1}}$

$= ba$

$a^3b^3 = (ab)^3$

$aaabbb = (ab)(ab)(ab)$

$a^{-1}aaabbbb^{-1} = \cancel{a^{-1}(ab)}aba\cancel{bb^{-1}}$

$= baba$

$a^2b^2 = (ba)^2$ ← for commutative

Cannot assume that it works for other.

② $a^4b^4 = (ab)^4$

Consider

$a^2b^2 = (ab)^2$

$a^2a^2b^2b^2 = a^2(ab)^2b^2$

$a^4b^4 = a^2b^2(ab)^2$

from ① $a^2b^2 = (ab)^2$

⇒ $a^4b^4 = (ab)^2(ab)^2$

$= (ab)^4$ ←

Works for only one ?.

③ $(ab)^3 = (ba)^3$ | Given that $a^3b^3 = (ab)^3$

$(ab)^3 = (ab)(ab)(ab) \overset{\text{from ③}}{=} a^3b^3 = aaabbb$

⇒

from

$a^4b^4 = (ab)^4$

$aaaabbbb = (ab)(ab)(ab)(ab)$

~~Prove a^3b^3 $(ab)^3 = (ba)^3$~~

$\cancel{a^4b^4 = (ab)^4}$

$\cancel{a^{-1}a^4b^4b^{-1} =}$

$\cancel{a^3b^3 = a^{-1}(ab)^4b^{-1}}$

$\cancel{a^3b^3 = (ba)^3}$

$a^4b^4 = (ab)^4$

$aaaabbbb = (ab)(ab)(ab)(ab)$

$\times (ab)^{-1}$

$\cancel{aaaabbbb(ab)^{-1} = (ab)^3}$

Prove that $(ab)^3 = (ba)^3$

$\cancel{(ab)^{-3}(ab)^3 = (ab)^{-3}(ba)^3}$

$\cancel{(ab)^3 = (ba)^3}$

$\cancel{(ab)^3(ba)^{-3} = e}$

$\cancel{aaabbb(ba)^{-3} = e}$

$a^4b^4 = (ab)^4$

$a^4b^4(ab)^{-1} = (ab)^3$

$(ab)^2(ab)^2(ab)^{-1}$

from ① $a^2b^2 = (ba)^2$

$(ab)^2(ab) = (ab)^3$

$a^2b^2 = (ba)^2$

$aa^2b^2b =$

$a^4b^4 = (ab)^4$

$\cancel{a^{-1}}aaaabbbb\cancel{b^{-1}} = \cancel{a^{-1}}(ab)(ab)(ab)\cancel{(ab)b^{-1}}$

$a^3b^3 = (ab)^3 = bababa$

$= (ba)^3 \longleftarrow$

Common method for showing Commutativity

Properties preserved by Isomorphism (Group Invariants)

- For infinite groups it is not practical to use group tables to prove isomorphism. as when proving that when 2 groups are not isomorphic we would need to check all elements. We therefore need to check group invariants.

Order of an element. $|G|$

- It is the number of elements present in a group.
- Isomorphism preserves order.

If $(G, \circ)$ and $(H, \ast)$ are isomorphic $\Rightarrow |G| = |H|$

→ Both groups could be finite (with same no. of elements) or infinite.

ex 1) Write down i) $|S(\triangle)|$ ii) $|S(R)|$ iii) $|S(\square)|$

Rectangle, Square

Deduce that no two of these groups are isomorphic.

i) $|S(\triangle)| \rightarrow$ Symmetry group of equilateral $\triangle$

↳ $= 6$

$|S(R)| = 4$

$|S(\square)| = 8$

No group have same order hence none are possibly isomorphic