Banach-Tarski Paradoxical Decomposition

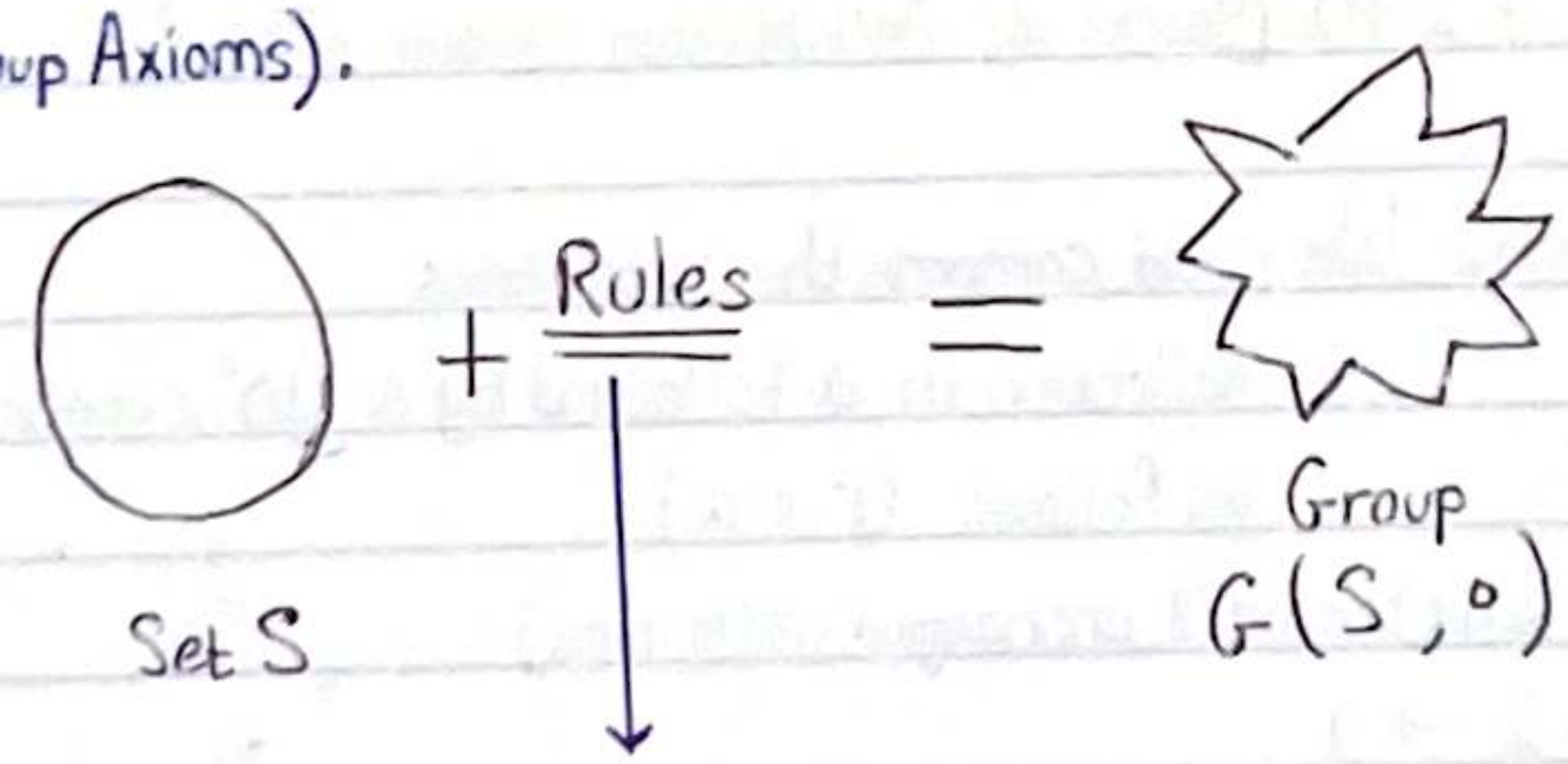
Requirements : Some Group Theory , Free Group
Paradoxical Decomposition of F_2.
Axiom of Choice (ZFC)
Group of 3D rotations isomorphic to the Free group in 2 generators.

History.
The term Group is derived from the French word 'Groupe' by Evariste Galois for the Study of the Symmetry Group of roots of polynomials.

Elementary Group Theory and Isomorphisms.

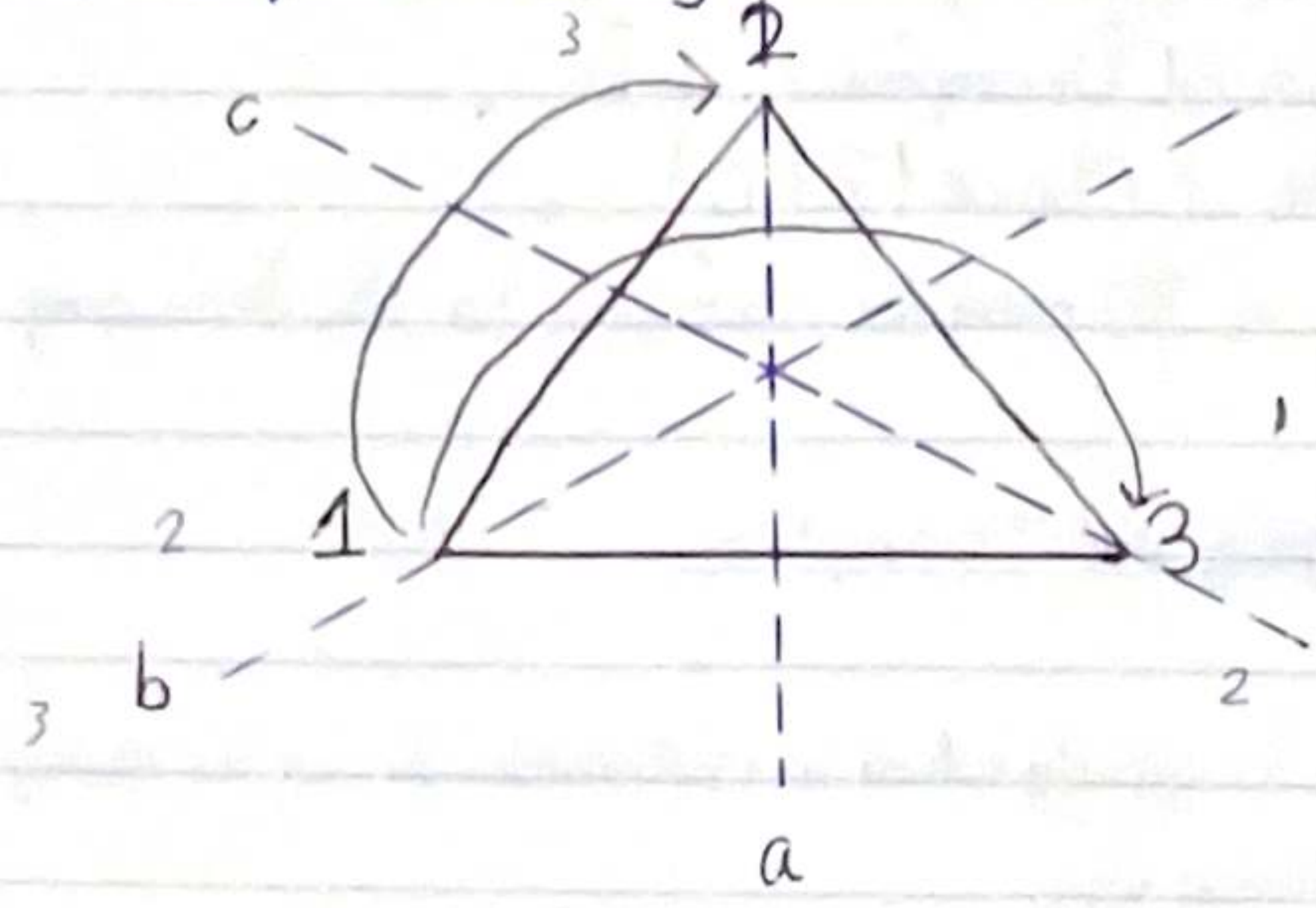
What is a group?
- This abstract object is greatly valued in mathematics due to its ability to encode the notion of symmetry in a fashionable way.

~~The Group is simply a set G~~
The typical Group G consists of a set of elements of which ~~&~~ any 2 elements can combine with each other under a certain binary group operation $\circ$ provided certain rules are followed (Group Axioms).



① Closure
② Identity
③ Invertibility
④ Associativity

Let's take a look at the D_3 (Dihedral group order 3) group which encodes the symmetric properties of an equilateral trianglel.



Rigid transformations applied to the figure leaving it "looking unchanged".

The numbers are present just to keep track of the locations and are distinct from the shape.

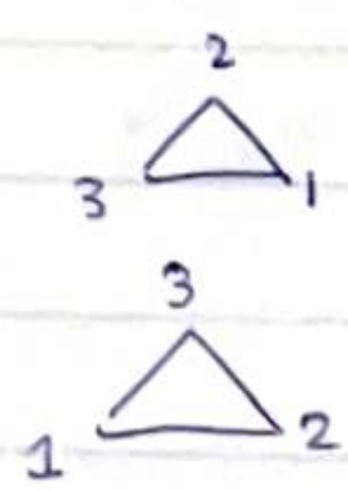
Possible symmetries
Reflection in axes a, b and C
- Clockwise rotation of 120° [Same as Anticlockwise rotation of 240°]
- Anticlockwise rotation of 120°
- Clockwise rotation of 240° [Same as Anticlockwise rotation of 120°]

~~All the elements present~~ We could compose the symmetries

e.g. Reflection in a followed by a 120° rotation c/w is denoted as follows $(r \circ a)$

$r \circ a$ sends 1 to 3 and leaves 2 unchanged (reflection)

rotation sends $3 \to 2 \to 1$



$(r \circ a) = b$ (Same as reflecting through b.)

We can construct a table to map all the transformations.

We also need an Identity.

Free Group.

- A group F would be 'free' if it satisfied no other conditions than the group axioms.

- Properties

Non-identity elements should have infinite order
[Group Axioms do not dictate the order of the elements]

- We do not let the nature of the elements and the result of their compositions influence our group!

- Non-Abelian (No extra commutative conditions imposed)
unless $x = u^k$ and $y = u^l$ for some $u \in F$, $k, l \in \mathbb{Z}$

- All Subgroups should be Free

F_2

Suppose a group G is generated by two elements, say $\{a, b\}$
G is generated by $\{a, b\}$ - every element of G can be expressed as a word in
$\{a, a^{-1}, b, b^{-1}\}$ ab, $a^{-1}ba$, $bbbbb^{-1}ab = b^4 b^{-1}ab$

G has an id $= 1$
must have aa^{-1}, bb^{-1}, $b^{-1}b$, $a^{-1}a = 1$ in G
word

String concatenation
?
$u = abab^{-1}$, $v = ba^{-1}a$
$uv = abab^{-1}ba^{-1}a = abaa^{-1}a = aba$
Reduction

Given any 2 words, concatenation & reduction always terminates and always give a unique reduced word.

Let $A = \{a, b\}$, F_A denotes the set of all reduced words on $\{a, a^{-1}, b, b^{-1}\}$

$(F_A, \circ)$ is a group with identity element ε (Empty word).

If G is a group and $\phi : A \to G$ is any map then the map $\phi' : F_A \to G$ defined by

$\phi'(x_1^{\varepsilon_1} \dots x_r^{\varepsilon_r}) = \phi(x_1)^{\varepsilon_1} \dots \phi(x_r)^{\varepsilon_r}$ is a group homomorphism

in set ± 1

in set A

Given any map ~~hh~~ from set A into any Group G, the map will extend into an uniquely group homomorphism.

Any group generated by 2 elements is a quotient of F_A.

Quotient Group

e.g. G/α

If α is reached, it is "divided" (Quotient out) to yield the identity of the group effectively getting rid of the element α

Since set has only 1 identity & no repetition, the element α is obliterated.

Formal Definition of a Free Group

Let X be a set.

Suppose F is a group and $X \subseteq F$. We say the group F is free on X if, given any group G and any map $\emptyset : X \to G$, $\emptyset$ extends uniquely to a group homomorphism $\emptyset' : F \to G$

$$\begin{tikzcd} X \arrow[r, "\emptyset"] \arrow[d, hook] & G \\ F \arrow[ur, dashed, "\exists!\ \emptyset'"'] & \end{tikzcd}$$

$\emptyset$ maps the elements of set X into G

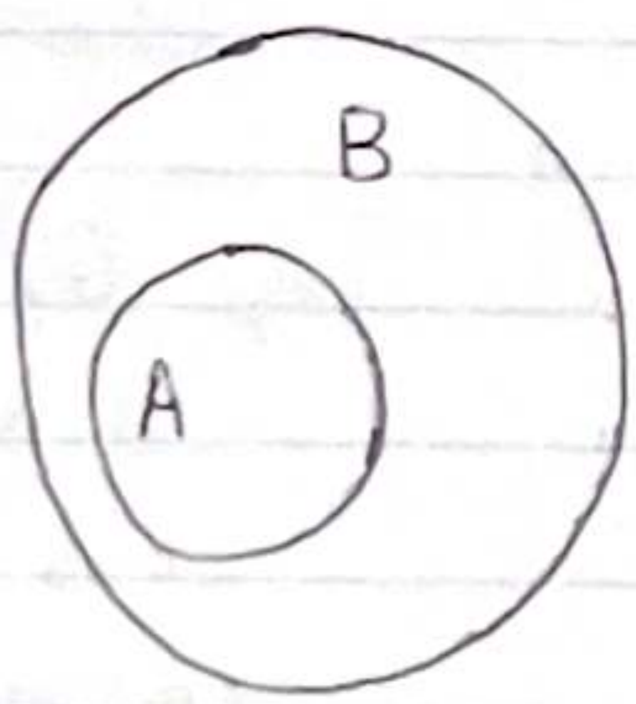
$x \hookrightarrow F$ (inclusion map) maps every element of

↘ The subset X of F

set X to itself in F

F is a free group iff we have unique group homomorphism $\emptyset'$ that extends $\emptyset$ s.t the diagram commutes.

<u>NOTES</u>



A is a subset of B ($A \subseteq B$)

B is a superset of A ($B \supseteq A$)

$\subseteq$: is a subset of

$\subset$: is a proper subset of

① Inclusion Map / Canonical Injection

- If $A \subseteq B$, then the inclusion map is the function f i that sends each element x of A to x, treated as an element of B.

$$i : A \to B, \quad i(x) = x$$

$$i : A \hookrightarrow B$$

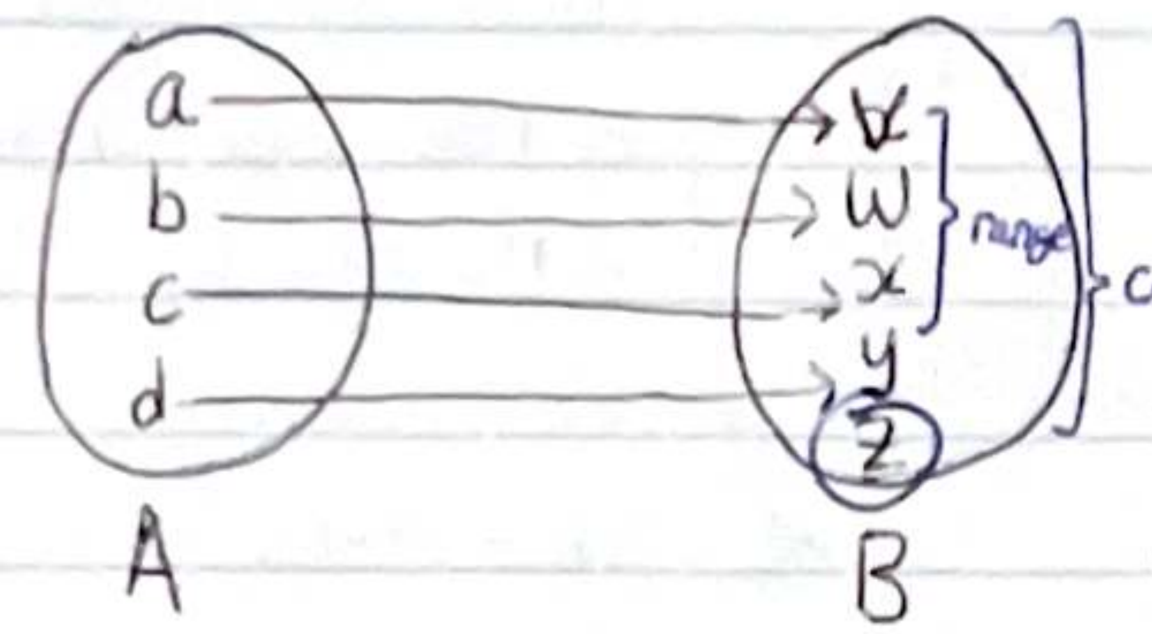
Structure Preserving Maps Between Mathematical Structures

Some properties of functions

$$\boxed{f : A \rightarrow B}$$

Defn of Injection: Every element of ~~range~~ range is linked to only one element of the domain.

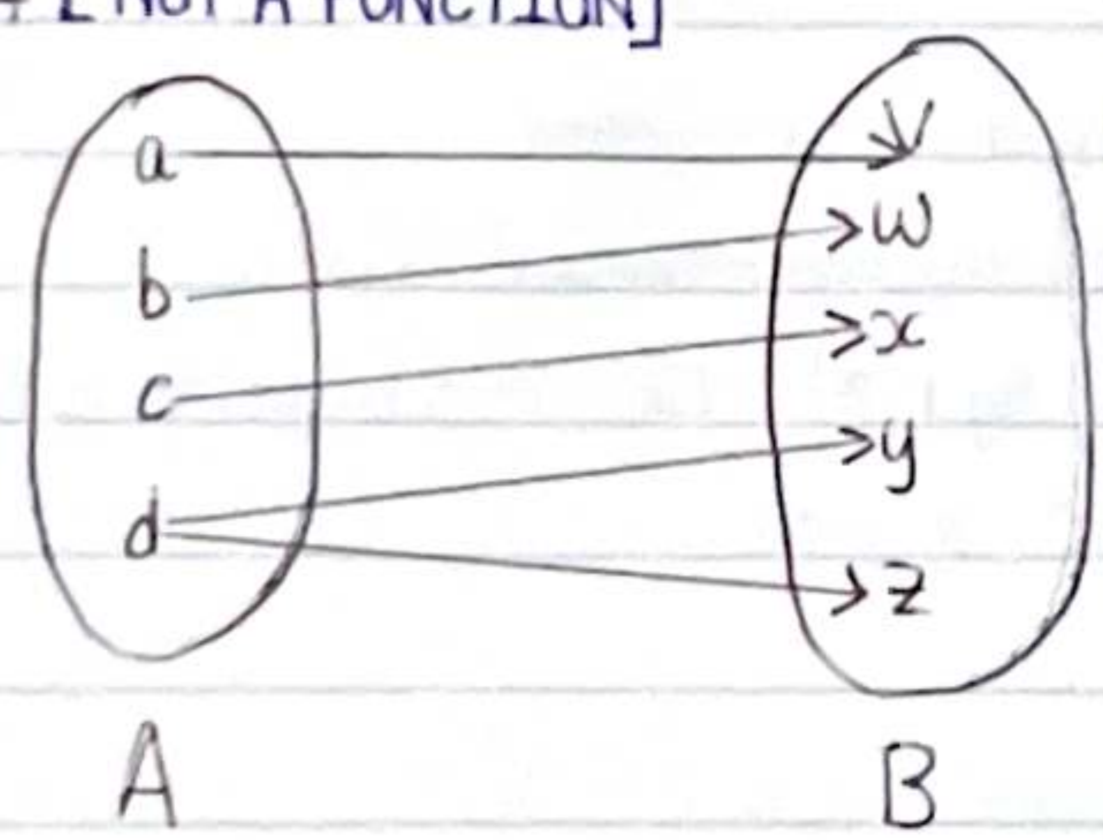
Injection



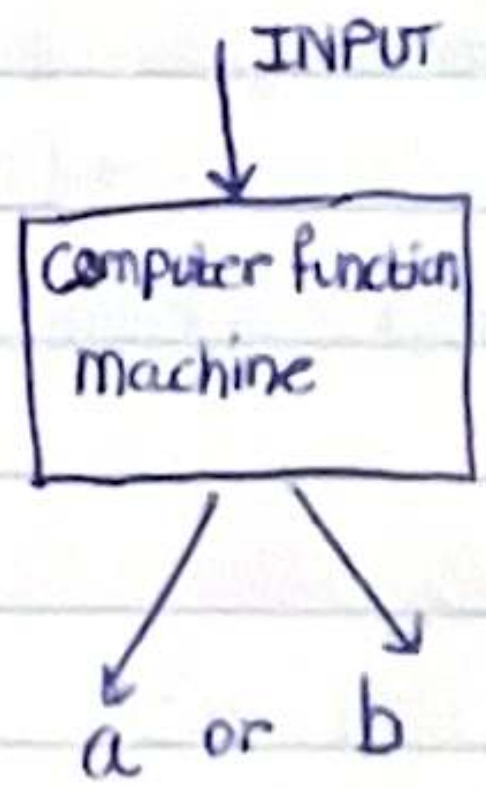
Injective as all elements of domain is linked to only one ~~element~~ element of the co-domain. X

It is not Surjective as an element of the co-domain was left behind!

~~Surjection~~ [NOT A FUNCTION]



This mapping is not a function as 2 elements in the co-domain are assigned to one element of the domain.



Note: Use the vertical line test to know if it is a function or not and use the horizontal line test to know if it has an inverse or not!

Machine is useless. Given ~~an~~ one input, it produces 2 different results.

[More than one output for a given input]

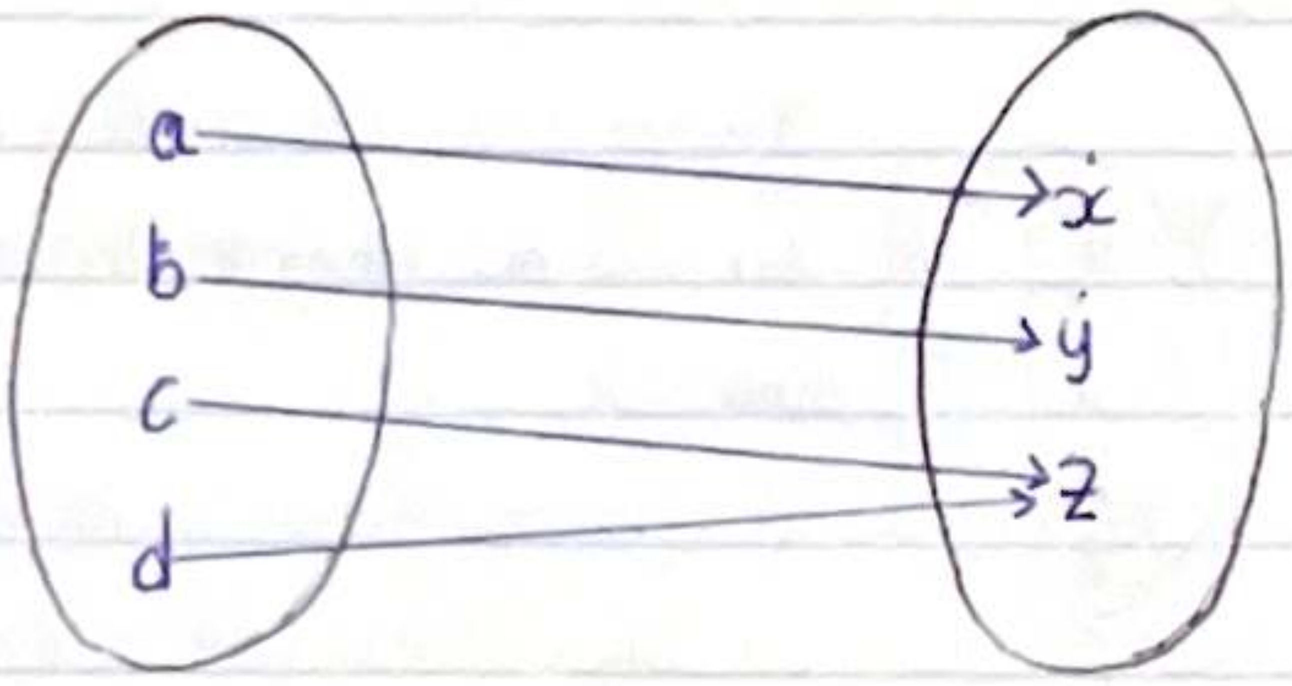
e.g. If we have a weather computer, we feed it data and it produces results of both storm and clear weather, it is useless.

Area of co-domain covered by function

Surjection

Target of the function

- Surjective if range equals the co-domain
- Surjective if every element of co-domain is assigned to at least one value in the domain or (no one in the co-domain is left behind).



It is Surjective but not injective as an element of the range is linked to more than one element of the domain.

It is still a function as 2 inputs can have the same results. But we will not be able to determine the inverse map.

Bijection is when a function is both injective and surjective.

Note: for a bijection, domain and co-domain must have same cardinality.

e.g. Let $f: [0,\infty) \rightarrow [0,\infty)$ be defined by $f(x) = \sqrt{x}$. This function is both injective & surjective. Therefore, f is bijective.

Example 2

Suppose $f(x) = x^2$.

If $Dom(f)$, $coDom(f) \in \mathbb{R}$, f is neither injection nor surjection.

It is not surjection as range is not equal to coDomain.

$Img(f) = (\mathbb{R}^{+} \cup \{0\})$ Same as image (Output of the function - set)

$coDom(f) = \mathbb{R}$

No number in domain with image of -1 ~~$\forall x\ f(x) \neq -1$~~ $\boxed{\forall x \nexists y : f(x) = -y}$

f cannot go to -ve real numbers

It is not an injection as more than one distinct element in the domain is mapped to the same element of the range. e.g. $\left.\begin{matrix} f(1) = 1 \\ f(-1) = 1 \end{matrix}\right\}$ same!

eg. Prove that the function $f: \mathbb{N} \rightarrow \mathbb{N}$ be defined by $f(n) = n^2$ is injective.

Proof

Let $a, b \in \mathbb{N}$ s.t $f(a) = f(b)$
$\Rightarrow a^2 = b^2$ by defn of f
$\Rightarrow a = \pm\sqrt{b^2}$
$\Rightarrow a = +b$ or $a = -b$
Since $Dom(f) \in \mathbb{N} \Rightarrow$ both a, b must be non-negative
$\Rightarrow a = +b$

This shows $\forall a \forall b [\underbrace{f(a) = f(b)}_{\text{one-to-one mapping}} \rightarrow a = b]$ which shows f is injective

eg. Prove the function $g: \mathbb{N} \rightarrow \mathbb{N}$, defined by $g(n) = \lfloor n/3 \rfloor$ is Surjective.

Defn of Surjectivity : $Img(g) = coDom(g)$

We feed any arbitrary value x into g and shows that it holds. $\Rightarrow$ Surjection.

Let $n \in \mathbb{N}$
Notice that $g(3n) = \lfloor (3n)/3 \rfloor = n \ (\in \mathbb{N}) \Rightarrow \underbrace{g(x) \in \mathbb{N}}_{\text{Range of } g \text{ Shown}} \ \forall x \wedge coDom(g) \stackrel{def}{=} \mathbb{N}$

Since $3n \in \mathbb{N}$, this shows that n is in the range of g. $\Rightarrow$ Surjectivity.

Review

Proof Techniques

① To prove a function is injective/one-to-one we must show the following:

$f : A \to B$

~~$\forall x \in A$~~ $\left(\forall x,y \in A\left((x=y) \Rightarrow (f(x)=f(y))\right)\right)$ or contrapositive equivalent

$\forall x,y \in A\left((x \neq y) \Rightarrow ((f(x) \neq f(y))\right)$

To prove this statement, we begin by choosing two arbitrary elements of the domain and assume the hypothesis of the implication i.e. we begin with "Let $x,y \in A$ and assume $f(x)=f(y)$" we then use the rules of algebra to show the conclusion must follow.

e.g. Let $f:[0,\infty) \to [0,\infty)$ be defined such that $f(x)=\sqrt{x}$. Prove f is an injection.

Defn of Injection : $\forall x,y \in Dom(f)\left((x=y) \Rightarrow (f(x)=f(y))\right)$

from $f(a)=f(b)$ We obtained $a=b$ which is in accordance with the rules of algebra, ~~the~~

Consider $a,b \in Dom(f)$ i.e. $[0,\infty)$ and by defn of injection $f(x)=f(y)$

$\Rightarrow \sqrt{a} = \sqrt{b}$

$\Rightarrow a = b$ (Square both sides) → Square values are always +ve

This shows that $\forall a,b \in [0,\infty)\left(f(a)=f(b) \Rightarrow a=b\right)$

$\Rightarrow f$ is an injection.

We assumed that f was injective and reached a conclusion.
~~the conclusion in accordance~~
This conclusion and its hypothesis match the definition of injection (Theorem).
This conclusion has been reached based on the defined rules of algebra (axioms) (or other lemmas and Theorems - Their conclusions & hypothesis)
This is the proof!

We look at our statement and try to make it match the theorem definition using pre-existing rules (axioms)

Using Real number axioms

Prove that $\forall x \in \mathbb{R}, x^2 \geq 0$

We could define a function $f : x \mapsto x^2$ and ~~f~~: show that $(-\infty, \infty) \to [0, \infty)$

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② To prove a function is surjective, onto, we must show $f(A) = B$, we must show the 2 sets $f(A)$ and B are equal.

$f : A \to B$ — $\mathrm{Img}(f)$ — $\mathrm{coDom}(f)$

We assume the function f is well defined $\Rightarrow$ We must show $B \subseteq f(A)$

we know

$f(A) \subseteq B$ is a well-defined function

We could use the defn of a subset

Show every element B is also an element of A.

We begin proof by fixing an arbitrary element of B. We then use other defn, theorem, algebraic manipulation to show that this arbitrary element must be the image of some element of A.

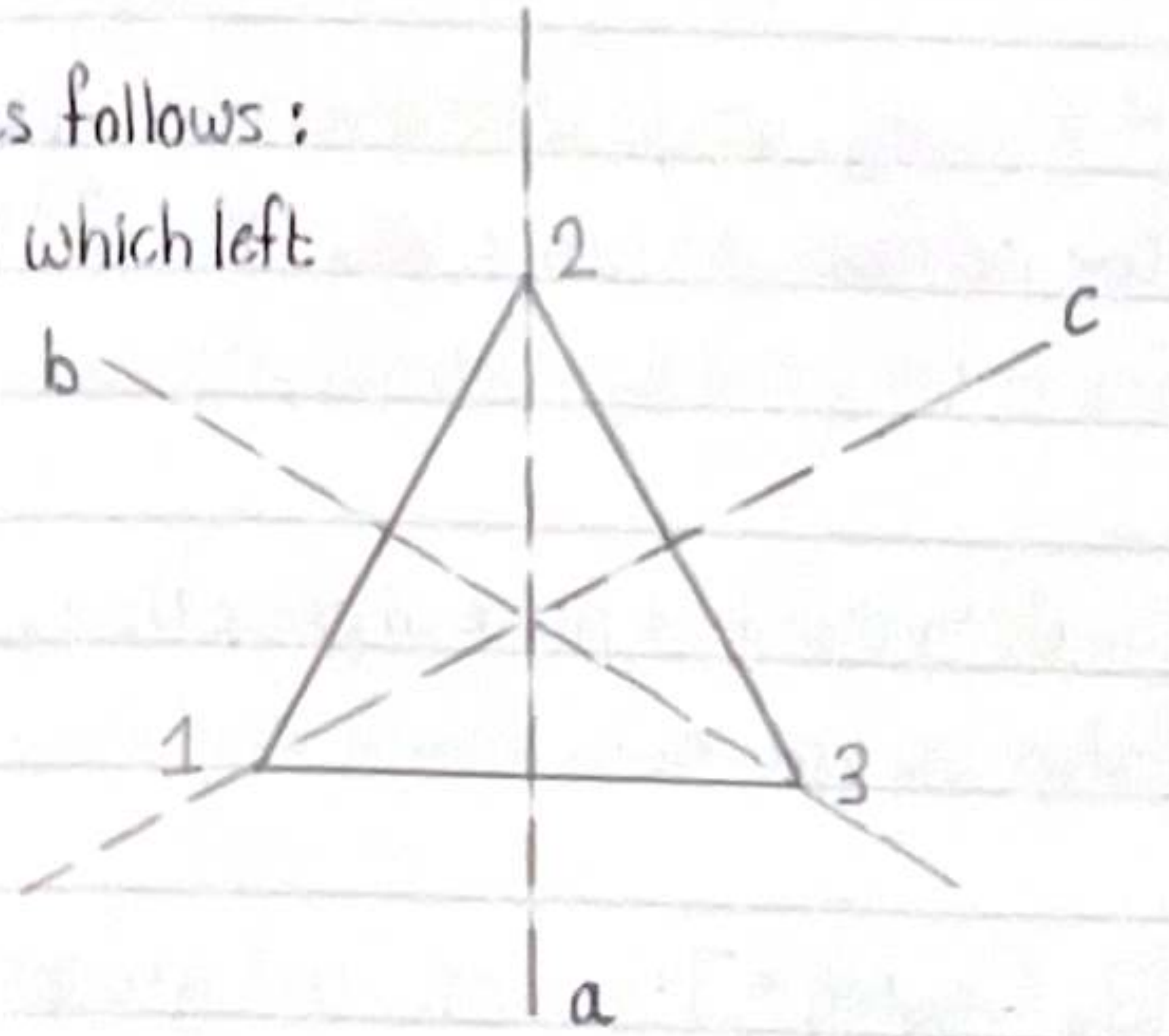
For any set X, there is a free group on X.

Def$\underline{\text{n}}$ of free group on X.

Suppose F is a group and $X \subseteq F$. The group F is free on X if, given any group G and any map $\phi : X \to G$, ϕ extends uniquely to a group homomorphism $\phi' : F \to G$

Group Actions

We were first introduced to a group as follows:
Six ~~set~~ possible operations were defined which left the △ looking similar.



e = identity (= 360°)
r = 120° c/w (= 240° cc/w)
s = 240° c/w (= 120° cc/w)
a = reflection in a
b = reflection in b
c = reflection in c

By composing the transformations, we identified a group behavior ~~and~~ (e.g. $a \circ r = b$) with the results of the compositions and recorded it in a Cayley Table.

$\circ$	e	r	s	a	b	c
e	e	r	s	a	b	c
r	r	s	e			
s	s					
a	a					
b	b					
c	c					

Now, let's consider the reverse case.

Galois Theory

Historically, groups were ~~develo~~ developped to study the properties of the roots of polynomials. The permutation group was one of the first groups developed with their closure, identity, invertibility and associativity properties.

With further development in group theory, it turned out that symmetries of shapes and other operations in modular arithmetic also had group behavior.

Isomorphisms could be shown between any 2 groups e.g. D_8 & $(\mathbb{Z}_8, +_8)$ or matrices & transformation of planes

By Cayley's Theorem, all groups are subgroups of the permutation group on n symbols.

Instead of choosing 2 objects and showing Isomorphisms between their group structures, we could make the permutation group "act" on these objects.
This would yield the same group by the Isomorphism implied by Cayley's Theorem.

Definition of a Group Action.

Let G be a group. Let X be a set. We say that G acts on the set X if there exists a map

$$G \times X \longrightarrow X$$
$$(g, x) \rightarrow gx$$

For the map to exist, (Surjection requires same cardinality), G (which could be a subgroup of a larger group) must have the same order as the cardinality of X.

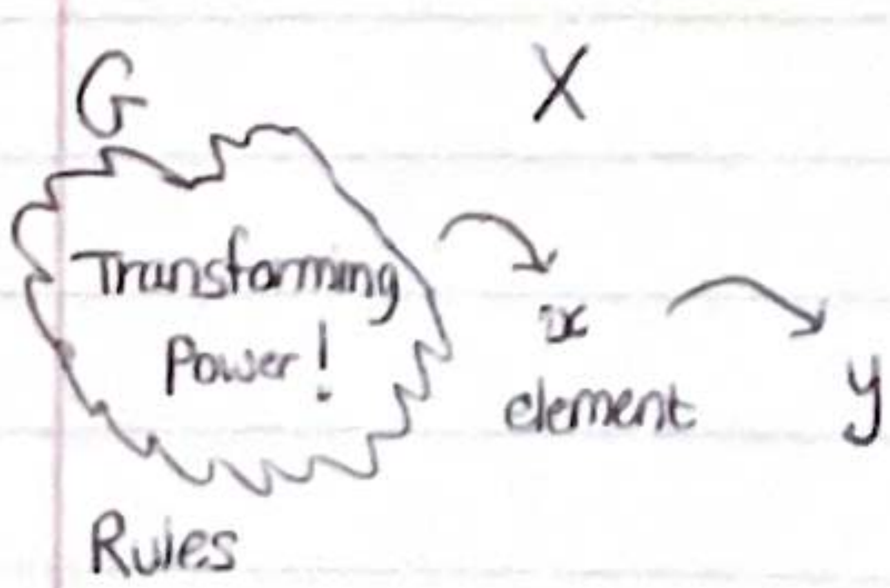
or could it be an injection only with some elements X left behind?

Such that

i) $ex = x \ \forall x \in X$

(ii) $g(hx) = gh(x) \ \forall g, h \in G \ \forall x \in X$

The Cartesian product will create every possible combination of G and X ensuring that G has effect on all elements of X.



The map then outputs which element of X is the result after G has operated on the element

Suppose we have $(\mathbb{Z}, +_6)$

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

→ Abelian (Symmetric about diagonal of Cayley Table)

~~Let's choose the Subgroup a subset {0,1,2} with same~~

$|(\mathbb{Z}, +_6)| = 6$

$\Rightarrow$ Order of Subgroups = 2,3 (Lagrange's Theorem)

$\{0, 3\}$ $\qquad$ $\{0, 3, 6\}$

Trivial Grap

half of the order of the group.

$+_6$	0	3
0	0	3
3	3	0

$+_6$	0	3	6
0	0	3	6
3	3	0	3
6	6	3	0