The 1-D Heat Equation

① Heat (or thermal) energy of a body with uniform properties, Q

$$\boxed{Q = mc\,\cancel{\theta}\,U} \leftarrow \text{Temperature}$$

Heat Energy

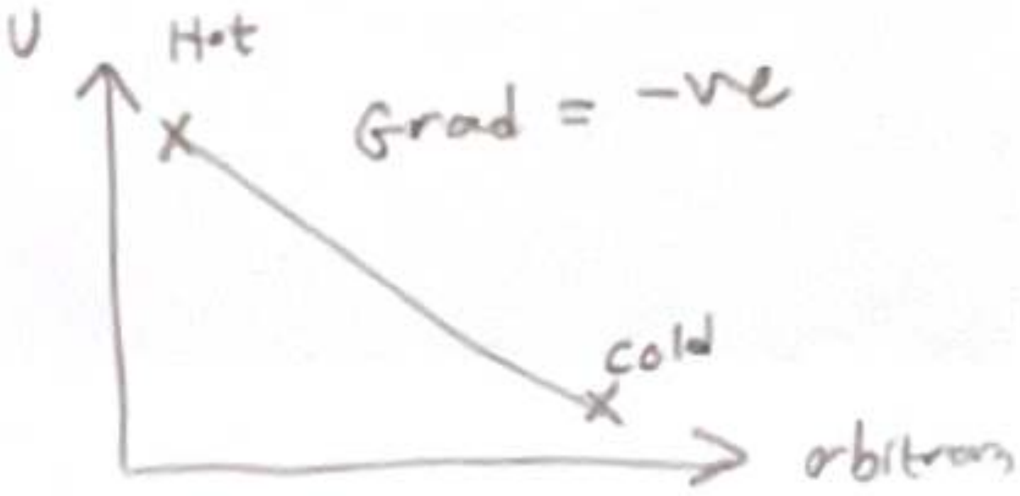
mass

Specific heat capacity

② Fourier's law : Rate of heat transfer is proportional to negative temperature gradient.

$$\boxed{\frac{\text{Rate of heat transfer}}{\text{Area}} = -K_0 \frac{\partial U}{\partial x}} \leftarrow : K_0 \Rightarrow \text{thermal conductivity}$$

$$\text{Rate transfer} \propto -\frac{\partial U}{\partial x}$$



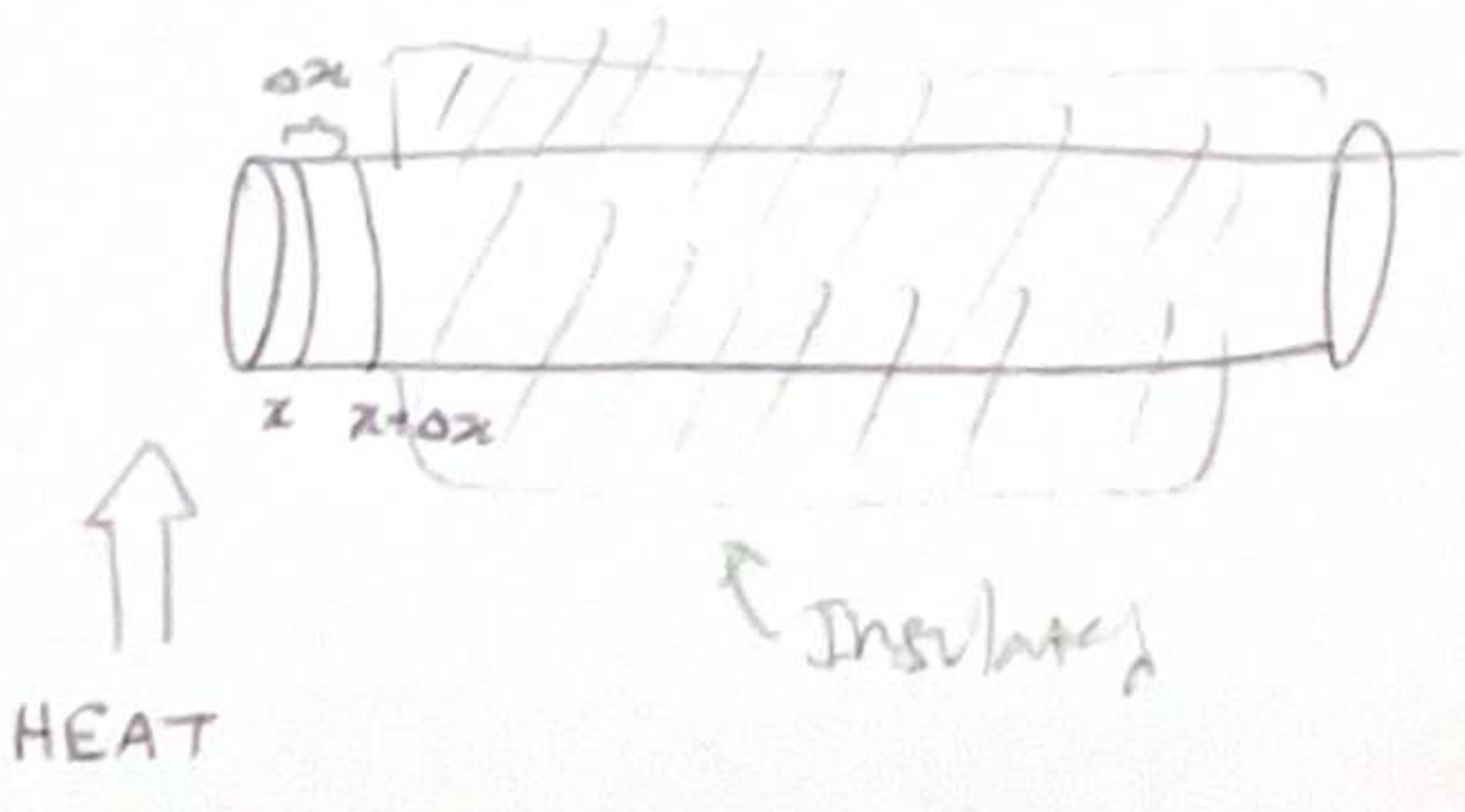
- Grad = +ve

⇒ Heat is transfered from hot to cold

③ Conservation of Energy.

Consider a uniform rod of length with non-uniform temperature lying on the x-axis, from $x = 0$ to $x = \ell$. [Uniform ⇒ density ρ, Specific heat capacity & K_0 (thermal conductivity are Constants)].
& Cross-Sectional Area A.

Assume the sides of the rods are insulated & only the ends are exposed & that there is no heat Source within the rod.



Consider a thin slice arbitrary of the rod of the width Δx between x and $(x+\Delta x)$

The slice is so thin that the temperature ~~between~~ throughout the slice is $u(x,t)$

Thus, Heat energy of segment $Q_{\Delta x} = C \times \underbrace{\rho A \Delta x}_{m=\rho V} \times U$

$$= c\rho A \Delta x U(x,t).$$

By conservation of Energy, Change of heat energy of segment in time Δt $=$ [heat in from left boundary] $-$ [heat out from right boundary]

From Fourier's Law,

$$c\rho A x U(x,t+\Delta t) - c\rho A x U(x,t) = \Delta t A\left(-K_0 \frac{\partial u}{\partial x}\right)_x - \Delta t A\left(-K_0 \frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

$$\frac{U(x,t+\Delta t)}{\Delta t} = \frac{K_0}{c\rho}\left(\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right)$$

Taking limits Δt as $\Delta x \to 0$, gives

$$\boxed{\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}} \qquad K = \frac{K_0}{c\rho}$$

HEAT EQUATION

$\Rightarrow$ Since slice was arbitrarily chosen, equation is valid throughout the rod.