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Physics Paper 5

Lagrangian Mechanics

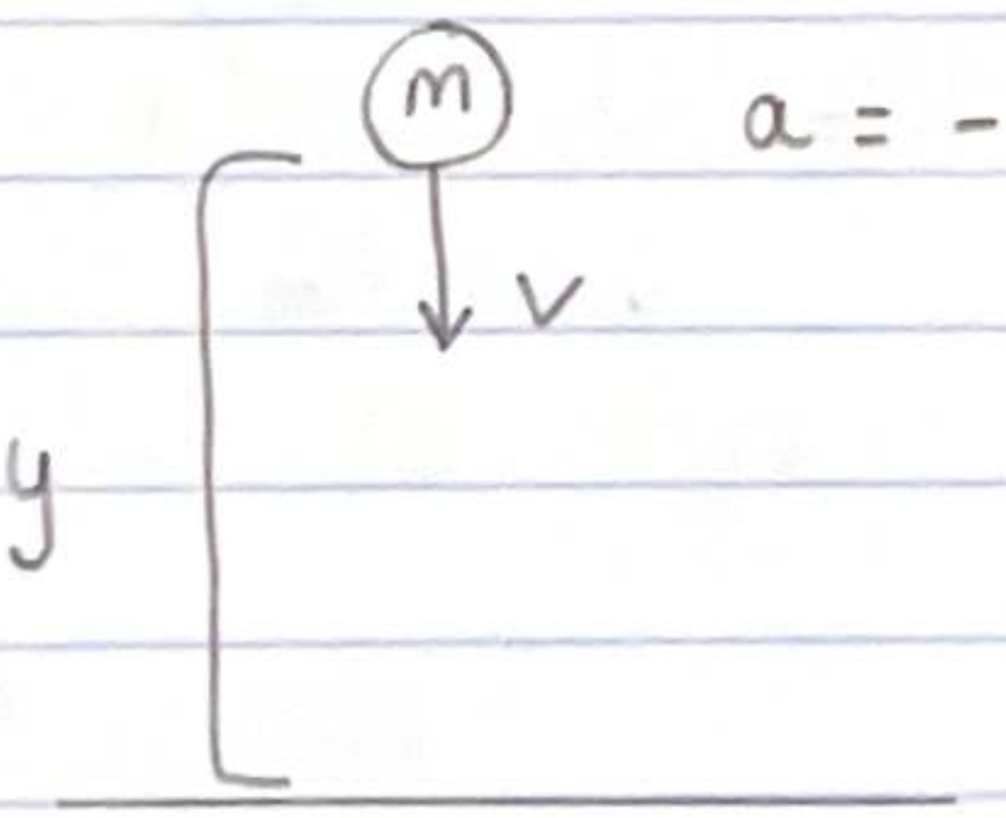
$$\text{Lagrangian} = L = KE - PE = T - V$$

Equation of motion can be determined by,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

Where $\dot{x}$ and x represent velocity and position vector respectively in generalised coordinates

$V = \dot{x}$ $\quad y = x$



$a = -9.81\ m/s^2$

$KE = \frac{1}{2}m\dot{x}^2 \qquad PE = mgx$

$$L = KE - PE = \frac{1}{2}m\dot{x}^2 - mgx$$

$$\frac{\partial L}{\partial x} = 0 - mg = -mg$$

$$\frac{\partial L}{\partial \dot{x}} = 2 \times \frac{1}{2}m\dot{x} = m\dot{x} - 0 = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$$

$$m\ddot{x} - (-mg) = 0$$ (put in general equation)

$$m\ddot{x} + mg = 0$$

$$\ddot{x} + g = 0$$

$$\ddot{x} = -g$$

(g is +ve
a is $-g$ in this case (-ve))

~~according to diagram, g is +ve~~

~~acc ($\ddot{x}$) is in direction of g, so it must be +ve~~

$\Rightarrow$ ~~$g = -9.81$~~

$\rightarrow$ $a = -9.81\, m/s^2$

PE = 0 when y = 0

When y increases PE increases

$\Rightarrow$ PE is +ve

PE = mgy

Since y is +ve above ground (reference point)

g must also be +ve)

$$KE = \frac{1}{2}m\dot{x}^2 \qquad PE = mgx$$

$$L = \frac{1}{2}m\dot{x}^2 - mgx$$

$$\frac{\partial L}{\partial x} = -mg \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$$

$$\Rightarrow m\ddot{x} - (-mg) = 0$$

$$m\ddot{x} + mg = 0$$

$$\underbrace{-mg}_{\text{weight}} = m\ddot{x} \qquad \Longrightarrow F = m\ddot{x}$$

Force of gravity (-ve)

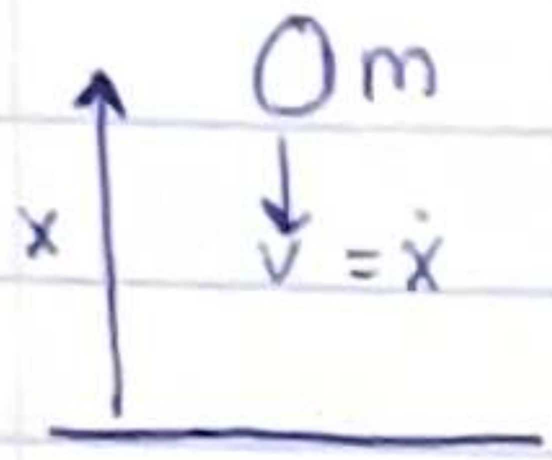
$$\boxed{F = ma}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$F = ma$$

Examples.

Free Fall.



$\rightarrow KE - EP = \frac{1}{2} m\dot{x}^2 - mgx = L$

$\frac{\partial L}{\partial x} = -mg \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$

$\Rightarrow m\ddot{x} - (-mg) = 0$

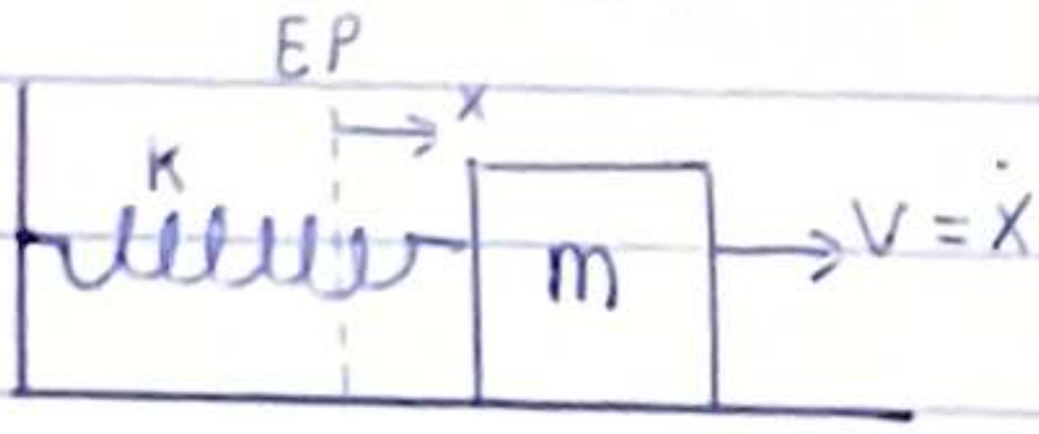
$m\ddot{x} + mg = 0$

$-mg = m\ddot{x}$

Not required to show $F = ma$

$m\ddot{x} = F = -mg$ $\quad (F = ma)$ Newton's 2^{nd} Law

SHM with Spring



$L = KE - PE = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$

units of L = J (energy)

$E = F \times d$

$\frac{E}{d} = F$

$\frac{\partial L}{\partial x} = -kx$ $\quad (F = -kx)$ Hooke's Law

$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$

$= m\ddot{x} + kx$ X

Coulomb's Law   Potential energy of q

Q (+)   (+) q

$$W = PE = qV = \underbrace{\frac{KQ}{R}}_{V}\, q$$

$$KE - EP = \frac{1}{2} m \dot{x}^2 - \frac{KqQ}{R}$$

Using general coordinates, $R = X$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{KqQ}{X} = \frac{1}{2} m \dot{x}^2 - KqQX^{-1}$$

$$\frac{\partial L}{\partial x} = 0 + (-1) - KqQX^{-2}$$

$$= \frac{KqQ}{X^2}$$

$$\left(F = \frac{Qq}{r^2} \times \frac{1}{4\pi\varepsilon_0} \right)$$ Coulomb's Law

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\downarrow \qquad\qquad \downarrow$$

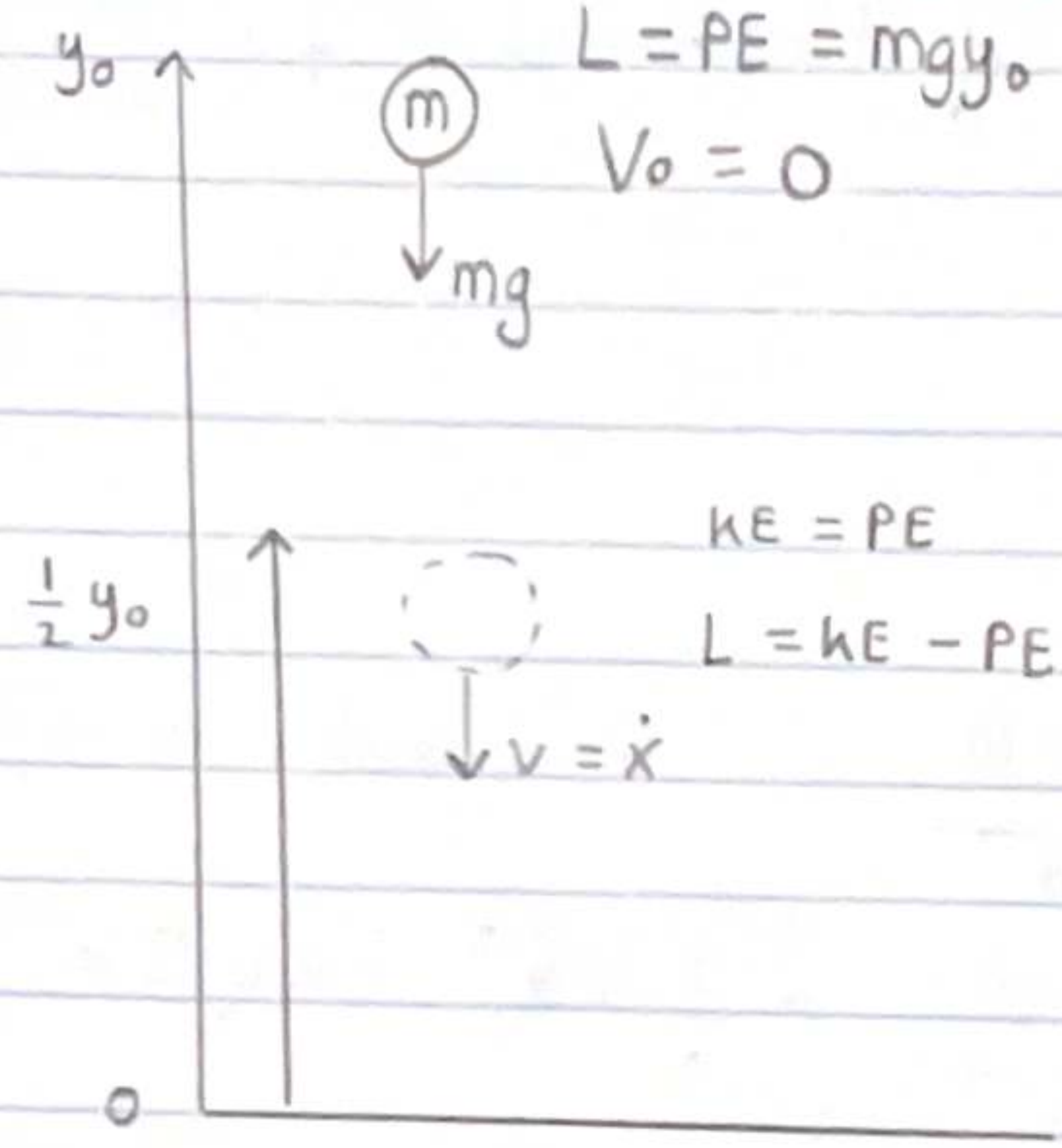
$$\frac{d}{dt}\left(\frac{E}{V}\right) \qquad \frac{E}{d} = F$$

$$\frac{E}{v \times \cancel{t}} \rightarrow d$$

$$\frac{E}{d} = F$$

Equations of motion from Lagrangian.

Free fall



$L = PE = mgy_0$ Since $KE = 0$ $L = -ve$ here

$V_0 = 0$

Since $KE = 0$

$L = -PE$

$KE = PE$

$L = KE - PE = 0$

Object is expected to proceed through simplest path.

$L = KE = \frac{1}{2} mv^2$ Since $EP = 0$ $L = +ve$

$$L = KE - PE = \frac{1}{2} m\dot{x}^2 - mgx$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial x} = -mg$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$-\frac{\partial L}{\partial x} = -mg \qquad \frac{\partial L}{\partial x} = mg$$

$\Rightarrow$ $\cancel{m\ddot{x} - (-mg) = 0}$

$\cancel{m\ddot{x} + mg = 0}$

$m\ddot{x} - mg = 0$ (Magnitude)

$m\ddot{x} = mg$

$ma = F$

$$\cancel{m}\ddot{x} = \cancel{m}g$$

$$\ddot{x} = g$$

$$a = g$$

$$\frac{dv}{dt} = g$$

$$dv = g\,dt$$

$$\int dv = g\int dt$$

$$v = gt + C$$

Initial conditions, $t = 0$

$$v = C$$

v at $t = 0$ $= V_0$

$$\Rightarrow V = gt + V_0$$

$$V = V_0 + gt$$

$$v = \frac{ds}{dt}$$

$$\frac{ds}{dt} = V_0 + gt$$

$$ds = V_0 + gt \quad dt$$

$$\int ds = \int (V_0 + gt)\,dt$$

$$S = V_0 t + \frac{gt^2}{2} + C$$

Initial conditions $t=0$

When $t=0$, S = initial position

$y = S$

$$y = V_0 t + \frac{1}{2} g t^2 + y(0)$$

$$y = y_0 + V_0 t + \frac{1}{2} g t^2$$ (Magnitude only)

$$m\ddot{x} - (-mg) = 0$$
$$m\ddot{x} + mg = 0$$
$$m\ddot{x} = -mg$$
$$\ddot{x} = -g$$

$$\frac{dv}{dt} = -g$$
$$\int dv = -g \int dt$$

$$V = -gt + C \qquad t=0 \quad V = V_0$$
$$V = V_0 - gt$$

$$V = \frac{dy}{dt}$$

$$\frac{dy}{dt} = V_0 - gt$$

$$\int dy = \int (V_0 - gt)\, dt$$

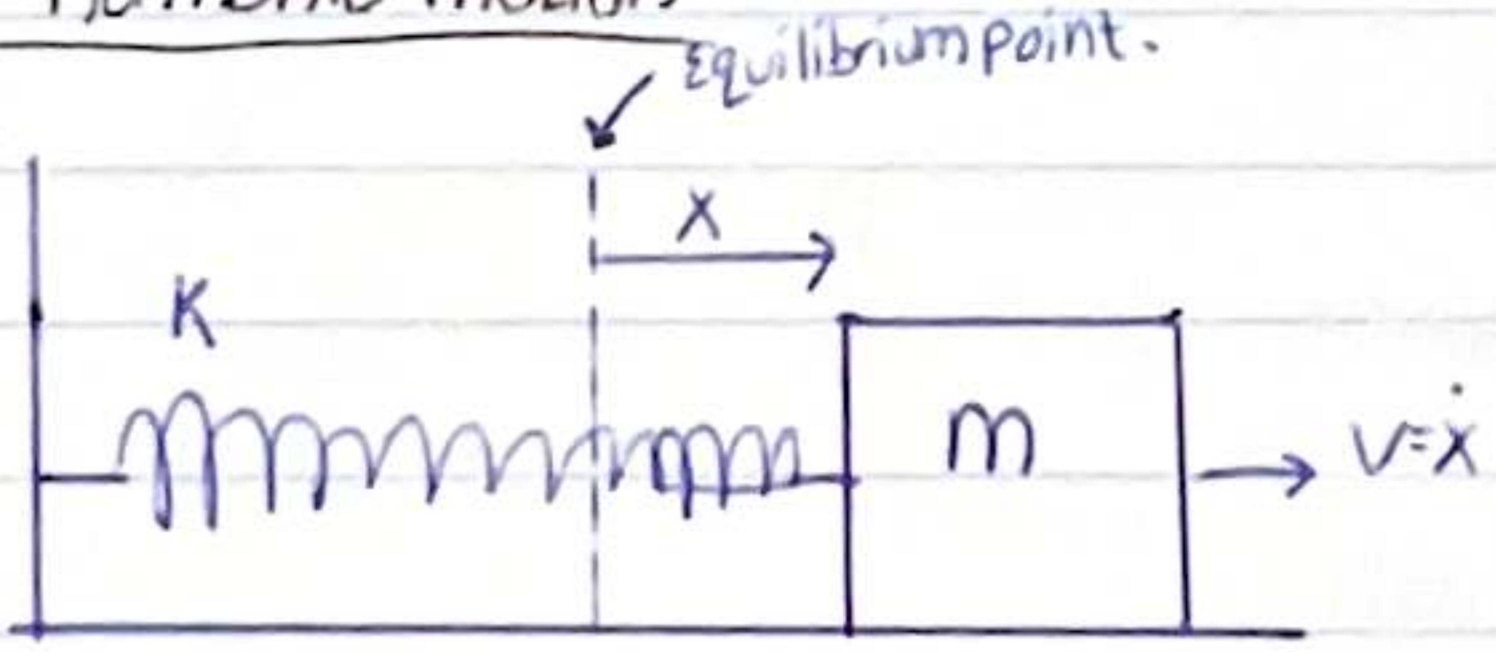
$$y = V_0 t - \frac{gt^2}{2} + c$$

Initial conditions, $t = 0$ $y = y(0)$

Initial position.

$$y = y_0 + V_0 t - \frac{1}{2} g t^2 \leftarrow$$

Simple harmonic motion



$$KE = \frac{1}{2} m v^2 \qquad PE = \frac{1}{2} K x^2$$
$$= \frac{1}{2} m \dot{x}^2$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2$$

$$\frac{\partial L}{\partial x} = -Kx \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} - (-Kx) = 0$$
$$= m\ddot{x} + Kx = 0$$

$m\ddot{x} + kx = 0 \quad (\div m)$

$\ddot{x} + \frac{k}{m} x = 0$

$\frac{d^2x}{dt} + \frac{k}{m} x = 0$ This has Soln in form of sin or cos

(Solve differential Equation)

$x = e^{mx}$

$\frac{dx}{dt} = me^{mx}$

$\frac{d^2x}{dt} = m^2 e^{mx}$

$m^2 e^{mx} + \frac{k}{m} (e^{mx}) = 0$

$e^{mx} \left(m^2 + \frac{k}{m}\right) = 0$

Auxiliary Equation

Consider

$\left(m^2 + \frac{k}{m}\right) = 0 \quad (\times m)$

$(m^3 + k) = 0$

$\frac{k}{m}$ represents angular frequency

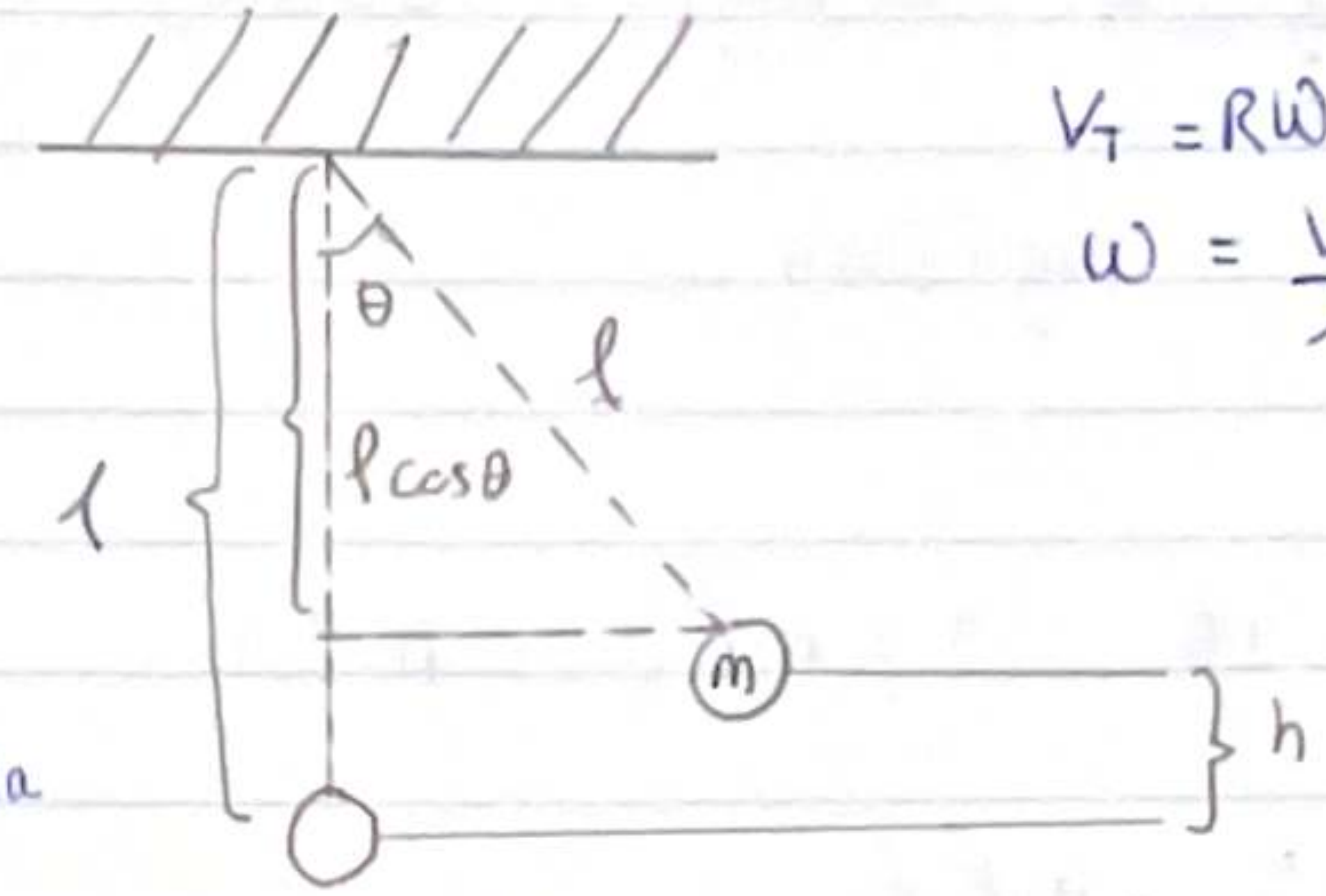
$\omega = \sqrt{\frac{k}{m}} \quad \omega^2 = \frac{k}{m}$

$\ddot{x} + \omega^2 x = 0$

~~$x = A($ $A\cos(\omega t)$~~

$A\cos(\omega t)$ or $A\sin(\omega t)$

Simple Pendulum



$V_T = R\omega = l\omega$

$\omega = \frac{V}{l}$

moment of inertia

$KE = \frac{1}{2} I\omega^2 = \frac{1}{2}(ml^2)\left(\frac{V^2}{l^2}\right)$

$= \frac{1}{2} mV^2$ Tangential Velocity

also

$\omega = \frac{d\theta}{dt} = \dot{\theta}$

$KE = \frac{1}{2} I \dot{\theta}^2 \leftarrow$

$h = l - l\cos\theta$

$PE = mgh = mg(l - l\cos\theta)$

$= mgl(1 - \cos\theta) \leftarrow$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

KE − PE

$$L = \frac{1}{2} I\dot{\theta}^2 - mgl(1-\cos\theta) = \frac{1}{2} I\dot{\theta}^2 - mgl + mgl\cos\theta$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} \quad (I = ml^2) = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\rightarrow ml^2\ddot{\theta} - (-mgl\sin\theta) = 0$$

$$= ml^2\ddot{\theta} + mgl\sin\theta = 0 \quad (\div ml)$$

$$= l\ddot{\theta} + g\sin\theta = 0$$

Since angle swept by pendulum is small,

$\sin\theta \approx \theta$

$$l\ddot{\theta} + g\theta = 0 \quad (\div l)$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \longrightarrow$$ (D.E has soln sin or cos)

$$\omega^2 = \frac{g}{l} \quad \omega = \sqrt{\frac{g}{l}}$$

$$\ddot{\theta} + \omega^2\theta = 0$$

$\theta(t) = \cancel{A\cos}\ A\sin(\omega t)$ or $A\cos(\omega t)$

sider $\cancel{\ddot{\theta} + \frac{g}{l}\theta = 0}$

$$\cancel{\ddot{\theta} = \omega^2 \theta}$$

$$\theta = A\sin(\omega t)$$

$$\omega = \sqrt{\frac{g}{l}}$$

$$\omega = 2\pi f$$

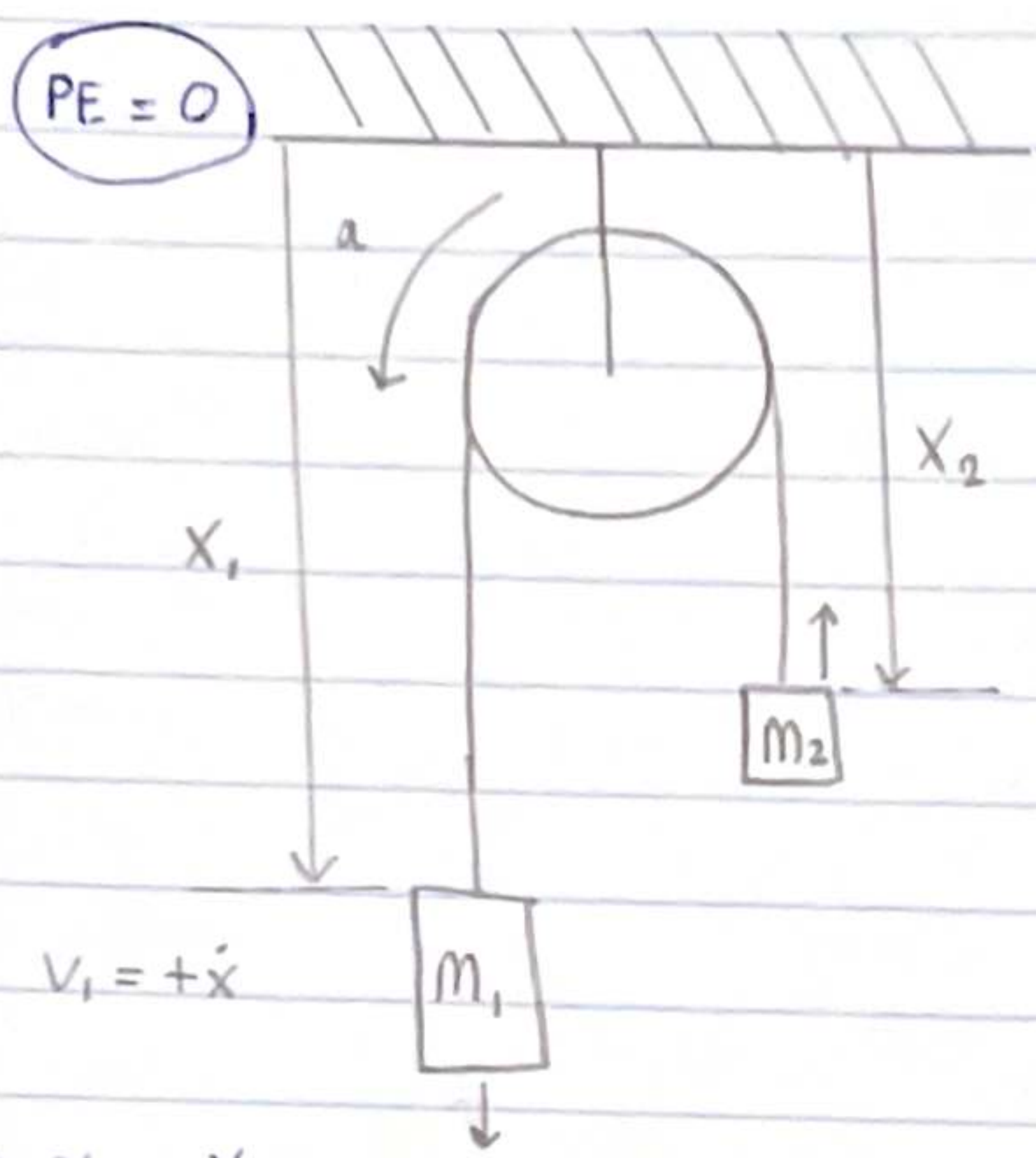
$$f = \frac{1}{2\pi}\omega$$

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

$$f = \frac{1}{T}$$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

The Atwood machine (pulley has no mass)



pulley has no mass.

$g = +9.81\ m/s^2$ $(\downarrow)$ +ve

in generalised x coordinate

$(+x) \downarrow$

$V_2 = -\dot{x}$

Let $X_1 = X$

$X_2 = \cancel{X}\, l - X$ where l is an arbitrary constant.

↑ the distance (downwards from top to bottom)

$$KE = T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2$$

$$PE = -m_1 g x - m_2 g (l - x)$$
$$= -m_1 g x - m_2 g l + m_2 g x$$

$$L = KE - PE = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 - (\overset{-m_1 g x}{\cancel{m_1 g x}} \mp m_2 g l + m_2 g x)$$
$$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + m_1 g x + m_2 g l - m_2 g x$$

$$\frac{\partial L}{\partial x} = m_1 g - m_2 g = (m_1 - m_2) g$$

$$\frac{\partial L}{\partial \dot{x}} = m_1 \dot{x} + m_2 \dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m_1 \ddot{x} + m_2 \ddot{x} = (m_1 + m_2) \ddot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$(m_1 + m_2) \ddot{x} - (m_1 - m_2) g = 0$$

$$(m_1 + m_2) \ddot{x} = (m_1 - m_2) g$$

$$\frac{m_1 + m_2}{m_1 - m_2} = \frac{g}{\ddot{x}}$$

or $\ddot{x} = \dfrac{(m_1 - m_2)}{(m_1 + m_2)} g$

$a = \left(\dfrac{m_1 - m_2}{m_1 + m_2}\right) g$ a is +ve Since $m_1 > m_2$

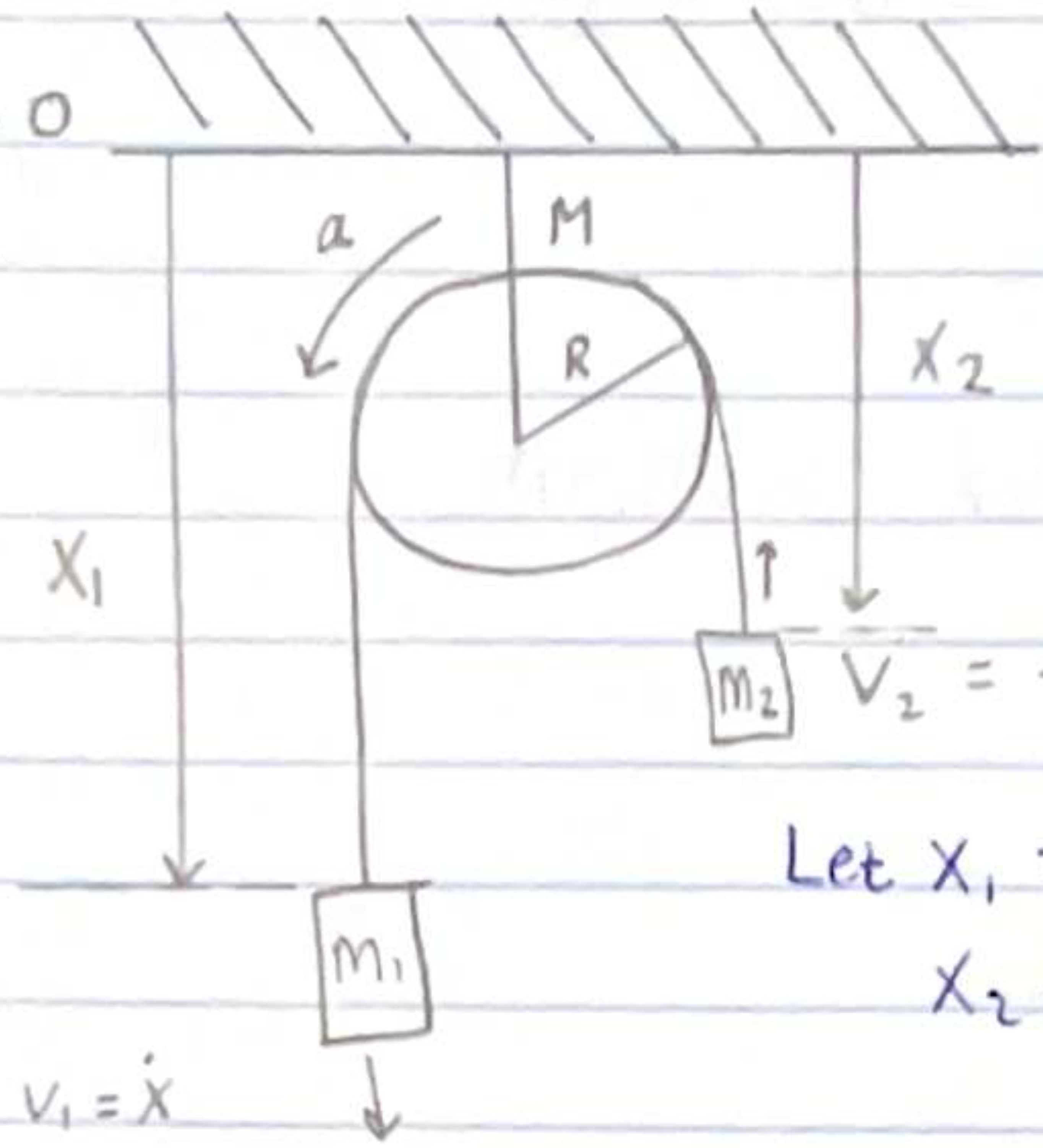
Atwood Machine (Pulley has mass) (Pulley has Moment of Inertia)

PE = 0

(PE down will be -ve)

$+x\ (\downarrow)$

$I = \frac{1}{2}MR^2$



$V_2 = -\dot{x}$

$v_1 = \dot{x}$

Let $X_1 = X$

$X_2 = l - X$ where l is an arbitrary constant

$g = +9.81\ m/s^2$

$$KE = T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}I\omega^2 \qquad \omega = \frac{V}{R} \quad \omega = \frac{\dot{x}}{R}$$

$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}M\cancel{R^2}\right)\left(\frac{\dot{x}^2}{\cancel{R^2}}\right)$$

$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{4}M\dot{x}^2$$

$$PE = V = -m_1gx - m_2g(l - x)$$
$$= -m_1gx - m_2gl + m_2gx$$

$$L = KE - PE = \cancel{\frac{1}{2}m_1\dot{x}^2 +} \left(\frac{1}{2}m_1 + \frac{1}{2}m_2 + \frac{1}{4}M\right)\dot{x}_2 - (-m_1gx - m_2gl + m_2gx)$$

$$= \left(\frac{1}{2}m_1 + \frac{1}{2}m_2 + \frac{1}{4}M\right)\dot{x}^2 + m_1gx + m_2gl - m_2gx$$