Tensor
Diagonalisation ✓
Conic section
Maclaurin series
general formula
Taylor series

Eigenvector and Eigenvalue

An eigenvector of a square matrix A is a non-zero column vector X such that $Ax = \lambda x$ where λ is a scalar, called the corresponding eigenvalue of A

eg.
$$\begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ -4 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -12 \\ -6 \end{pmatrix} = 3\begin{pmatrix} -2 \\ -4 \\ -2 \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} -2 \\ -4 \\ -2 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}$ with eigenvalue $\lambda = 3$

??
If we multiply an eigenvector by any scalar, then it is still an eigenvector ~~(though not the same)~~ ?? and the corresponding eigenvalue is unaltered.

eg. let x be an eigenvector of A with eigenvalue λ, then if α is any scalar,

$$A(\alpha x) = \alpha A x = \alpha \lambda x = \lambda(\alpha x) \leftarrow$$

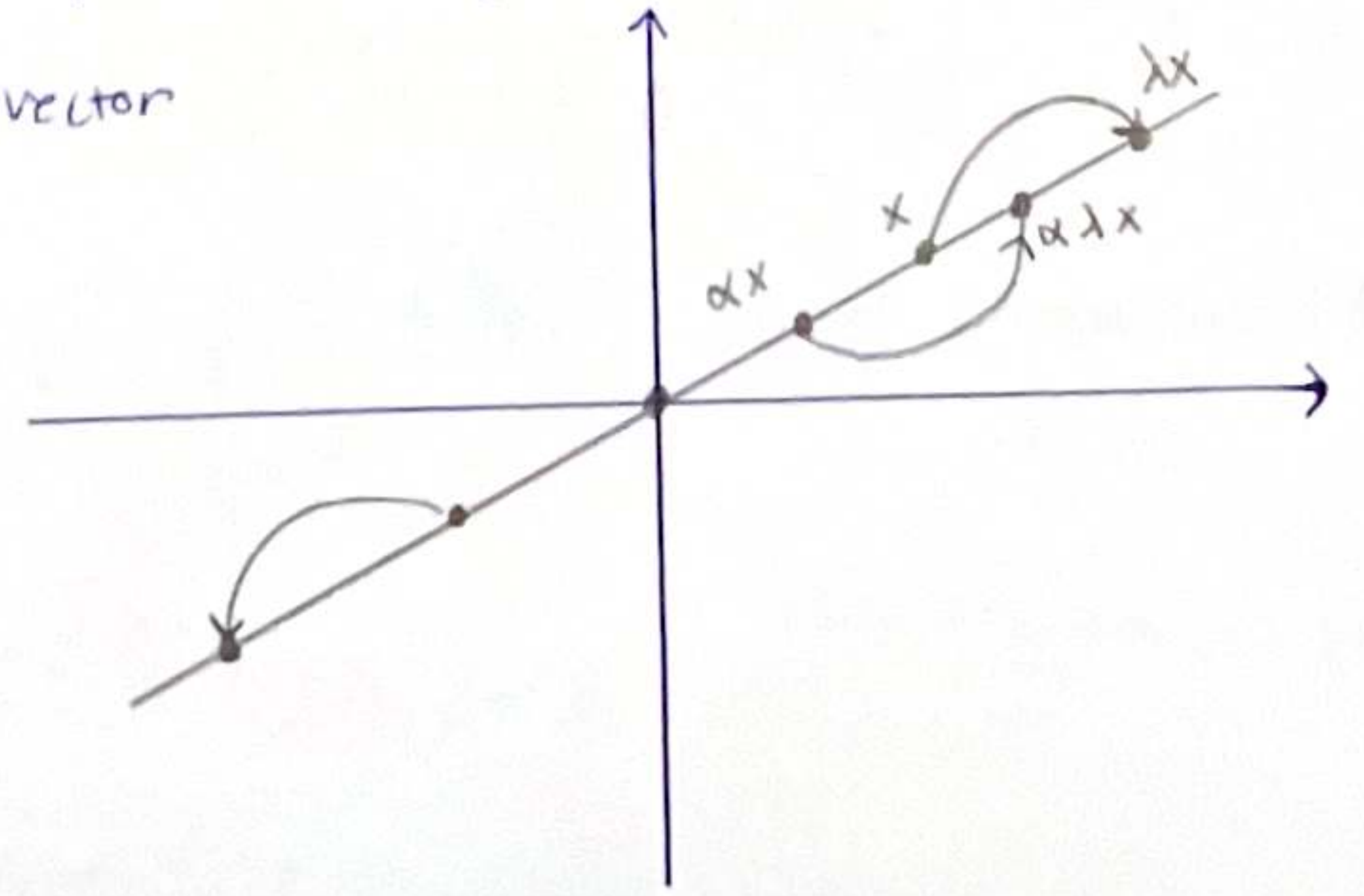
αx is also an eigenvector with eigenvalue λ.

Interpretation of eigenvectors in terms of transformation.

New value after transformation.
Matrix acts as scalar
multiple of A

Suppose x is an eigenvector of A then x is mapped to a scalar multiple of itself. ie. $(X) \times \hat{A} = A(X)$
In fact, any scalar multiple of x is mapped to a scalar multiple of x. In other words, the line through the origin passing through x is mapped onto itself. So, an eigenvector corresponds to a preserved line through the origin.

In other words, An eigen vector of a matrix is one when multiplied by ~~any~~ the matrix only cause a scalar change in the eigenvector. This ~~eigen~~ scalar corresponds to the eigen value λ. Eigen vectors are specific to a matrix, ie. there is no 'general' eigen vector



Finding the Eigenvalues of a Matrix.

$Ax = \lambda x$

Write $x = Ix$ (I is the identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$)

Collect all terms on L.H.S

$AX - \lambda I x = 0$

$(A - \lambda I)x = 0$ $\quad$ x: eigenvector
λ: corresponding eigenvalue.

eg. B is singular if and only if there is a non-zero vector x such that $Bx = 0$

x is non-zero

$\Rightarrow (A - \lambda I)$ is singular

$\Rightarrow |A - \lambda I| = 0$

Indicates matrix

Conversely, if $|A - \lambda I| = 0$, then there is a non-zero vector x such that $(A - \lambda I)x = 0$ $\Rightarrow Ax = \lambda x$
$\Rightarrow \lambda$ is an eigenvalue of A.

λ is an eigenvalue of A iff $|A - \lambda I| = 0$

The equation $|A - \lambda I| = 0$ is an equation in the Variable λ Called the characteristic equation of A.
Its roots are the eigenvalues of A.

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eg. Find the eigenvalues of the matrix : $A = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 1 \\ 3 & -\lambda \end{pmatrix}$$

$\Rightarrow$ the characteristic equation is : $\begin{vmatrix} 2-\lambda & 1 \\ 3 & -\lambda \end{vmatrix} = 0$

② find det. Characteristic equation

$\hookrightarrow (2-\lambda)(-\lambda) - (1 \times 3) = \lambda^2 - 2\lambda - 3 = 0$ $= (\lambda - 3)(\lambda + 1) = 0$
$\lambda = 3, \lambda = -1$

$\Rightarrow$ Eigenvalues are $\lambda = 3$ and $\lambda = -1$.

Note : unique Solutions are not expected as the Scalar multiple of a Solution is another Solution.

Finding Eigenvectors of a Matrix.

① First, we need to find the eigenvalues.

$$A = \begin{vmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{vmatrix}$$

$$A - \lambda I = \begin{vmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{vmatrix} - \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 0 & 1 \\ -1 & 2-\lambda & 3 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

find det. of $|A - \lambda I|$

$$2-\lambda \begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} - 0 \begin{vmatrix} -1 & 3 \\ 1 & 2-\lambda \end{vmatrix} + 1 \begin{vmatrix} -1 & 2-\lambda \\ 1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \left[(2-\lambda)^3 - 0\right] - 0 + \left[-(2-\lambda)\right] = 0$$

eigenvector Already given in 1st example before $= \begin{pmatrix} -2 \\ -4 \\ -2 \end{pmatrix}$

$$(2-\lambda)^3 - (2-\lambda) = 0$$

$$2-\lambda\left[\lambda^2 - 4\lambda + 3\right] = 0$$

$$(2-\lambda)(\lambda-3)(\lambda-1) = 0 \quad \Rightarrow \lambda = 1,\ \lambda = 2 \text{ and } \lambda = 3 \leftarrow \text{eigenvalues of } A.$$

Find the eigenvector of A corresponding to eigenvalues $\lambda = 1$ and $\lambda = 2$

When $\lambda = 1$, $A - \lambda I = \begin{pmatrix} 2-1 & 0 & 1 \\ -1 & 2-1 & 3 \\ 1 & 0 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$

Solve $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ for x, y and z

$$\Rightarrow x + z = 0 \quad -①$$

~~$y + 3z$~~

$$-x + y + 3z = 0 \quad -②$$

$$x + z = 0 \quad -③$$ → Redundant as = ① and 2 set of equations are needed to solve homogenous equations in 3 variables.

$\Rightarrow x + z = 0 \quad - ①$

$-x + y + 3z = 0 \quad - ②$

Suppose $x = k$

$\Rightarrow z = -k \quad$ - from ①

$-k + y - 3k = 0$

$\cancel{-k + y + 3k = 0}$

$y = \cancel{+2k}\ 4k \quad$ - from ②

Thus for any value of k,

$\underline{\underline{soln}}$: $x = k$

$\cancel{y = +4k}$ $y = 4k$

$z = -k$

ie. $\begin{pmatrix} k \\ +4k \\ -k \end{pmatrix}$ is an eigenvector of A with eigen value $\lambda = 1$

$\begin{pmatrix} 1 \\ +4 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} 2 \\ 8 \\ -2 \end{pmatrix}$ are eignvectors of A.

$\underline{\underline{\text{When } \lambda = 2}}$

$$A - \lambda I = \begin{pmatrix} 2-2 & 0 & 1 \\ -1 & 2-2 & 3 \\ 1 & 0 & 2-2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix}$$

Solve for x, y, z

$$\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow z = 0$

$-x + 3z = 0$

$x = 0$

$\Rightarrow x = z = 0$

So, y can take any value

$\Rightarrow$ and any vector of the form $\begin{pmatrix} 0 \\ k \\ 0 \end{pmatrix}$ eg $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ with

eigen value $\lambda = 2 \leftarrow$

Confirm for $\lambda = 3$

eg. Find eigenvectors of unit length for matrix

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$$ from ~~1st example~~ example before, eigenvalues : $\lambda = 3$, $\lambda = -1$

* for unit length $= 1$

for other value n, eg 3

$K^2 + K^2 = 3$

$\lambda = 3$: $A - \lambda I = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$

Solve, $\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$-x + y = 0$ $- ①$

$3 - 3y = 0$ $- ②$ $\Rightarrow$ $② = -3 \times ①$ $\Rightarrow$ redundant

Put $x = K$

$\Rightarrow x = y = K$
$y = K$

$\Rightarrow$ any vector in form $\begin{pmatrix} K \\ K \end{pmatrix}$ is an eigenvector of A with eigenvalue 3.

for unit length from eqn of circle ??, $x^2 + y^2 = 1$

we need $K^2 + K^2 = \underline{1}$ *

$2K^2 = 1$ $\Rightarrow K = \pm\sqrt{\frac{1}{2}}$ $\longrightarrow$ There are 2 possible soln for unit length for eigenvector of A with $\lambda = 3$

$\begin{pmatrix} \sqrt{1/2} \\ \sqrt{1/2} \end{pmatrix}$ and $\begin{pmatrix} -\sqrt{1/2} \\ -\sqrt{1/2} \end{pmatrix}$

when $\lambda = -1$: $A - \lambda I$ $= ~~\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \end{pmatrix}~~$

$= \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} \underset{+}{-} -1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$

Solve, $\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\left.\begin{aligned} 3x + y &= 0 \\ 3x + y &= 0 \end{aligned}\right\}$ Same eqn.

$y = -3x$

Put $x = K$

$\Rightarrow y = -3K$

$\Rightarrow$ any vector in the form $\begin{pmatrix} K \\ -3K \end{pmatrix}$ is an eigenvector of A with eigenvalue $= -1$

General solution $\downarrow$

For unit length, $K^2 + (-3K)^2 = 1$

$10K^2 = 1$ $\Rightarrow K = \pm\sqrt{\frac{1}{10}}$

~~Possible soln for unit length~~

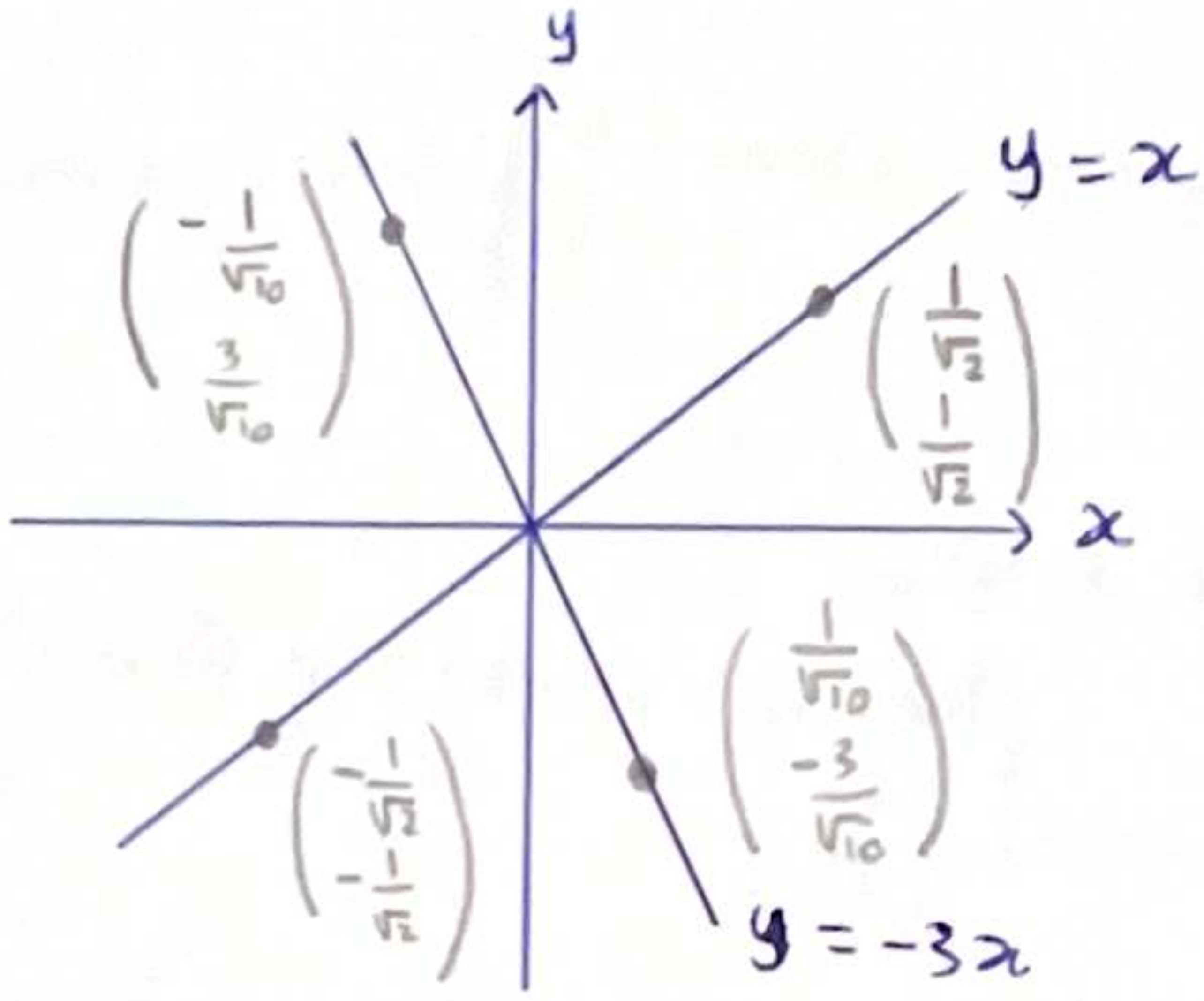
~~$\Rightarrow$ $\begin{pmatrix} \sqrt{1/10} \\ \sqrt{1/10} \end{pmatrix}$ and $\begin{pmatrix} -\sqrt{1/10} \\ -\sqrt{1/10} \end{pmatrix}$~~

$$x = K$$
$$y = -3K$$
$$K = \pm\sqrt{\tfrac{1}{10}} = \pm\frac{1}{\sqrt{10}}$$

Possible soln for unit length ~~of~~ for eignvector of A with $\lambda = -1$

$$\begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} \end{pmatrix} \text{ and } \begin{pmatrix} -\frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix}$$

The two eigenvectors in this example correspond to two preserved line through the origin namely, $y = x$ and $y = -3x$



Diagonalisation

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow$$ For Each row acting on $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ we are reducing 1 dimension.

Row 1 is the x component of all the basis.

The matrix itself represents the transformation. → It is the expression of LT in Coordinates

The transformation has rows & columns. → Basis change involved in transformation

rows → Covector measurements involved in that transformation.

In the coordinate representation with a matrix, the columns are what the transformation maps the basis to.

They are the image of the bases

The rows represent the n^{th} coordinate of the bases image.

[They represent what the ~~bases~~ x coordinates of the bases become under the transformation.]

We use that information to know how the input vector influences the x-coordinate of the output we see.

(By linearity, transformation of input space basis to image basis will have same effect on input vector)

The ~~Second~~ x coord. of first basis combines with first component of input.

Second basis with Second and so.

→ This is because the Second component y is actually $y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ → 2^{nd} original basis

Each one we do drops one dimension

~~It is projecting the new basis on the old one coordinate wise~~

The co-vector for a coordinate measures how much the basis change had an effect on that coordinate ~~during transformation from domain to image~~ → it sums all accumulations on how the basis change can affect ~~that~~ new coord. and scales that change with LC.

x-coord of ~~the~~ 1^{st} basis combines with x of 1^{st} basis

2^{nd} basis combines with y of 2^{nd} basis

3^{rd} basis combines with z 3^{rd} basis

and then rescale via linear combinations.

$\boxed{\text{The rows of the matrix are linear functionals.}}$

Each row corresponds to a component of the ~~codomain~~ set basis of the Image. → $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

The Column represent where each basis gets mapped to.

The linear functional as row of a matrix to get the first component of output combines the first component of the first basis with the first component of the input vector, the first component of 2nd basis with 2nd component of input, the first component of the 3rd basis with the third component of input all that to generate the first component of output

What can I do with all the x components of the Image basis?

Why am I dotting my vector with all the x basis component.

What do all the x-components of the basis represent?

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

~~To get the x component of our vector in the new image we project $\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{Dom}$~~

It's not a vector; it's a dual.

To get the i^{th} Coordinate of the output, I collect all the i^{th} coordinates of the bases and dot it with the whole input.

It is not a vector.

It is a linear functional specified by the i^{th} coordinates of the bases to extract the i^{th} coordinate in an input.

~~The first y~~ Pair with

The old y & z pair with the x components of 2^{nd} & 3^{rd} column basis

It is not a dot Product / Inner Product. It is the action of a covector on a Vector.

↳ The linear functional is formed from the ~~ith~~ i^{th} coordinate of the bases image. It extracts the i^{th} coordinate of any input after transform.

It needs to read all components not only the x component to produce the new x coordinate.

The old y or z might contribute to the new x

$\boxed{A\vec{x} = \vec{b}}$ $\boxed{T: V \to W}$ $\boxed{\text{Projection \& Rejection}}$

First View → What was I doing?

Wanted to move/transform change from one state to another. → Tempted by the image I was shown.

① I follow the columns & Scale myself accordingly. → See myself in the image.
↘ Form part of beauty.

② What if I could have better? what if I get sent to the world
Need to fulfill the need to measure. → Measure with rows
↘ Get me component by component.

Loss in so much information.
Forgot what I was chasing.
Forgot Greater picture.

Expand on hyperplanes here.
↳ More details into splitting me piece by piece.

The matrix has nothing to do with the x,y,z of the vectors
It is just a collection of rows & columns.

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot x + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \cdot y + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \cdot z$$