Probability.

Probability Measure

Probability Space $(\Omega, \mathcal{F}, \mathbb{P})$

Sample Space

Event Space
(σ-algebra)

- The Sample Space Ω is the set of all possible outcomes of an experiment ($\Omega \neq \emptyset$)
  e.g. $\{H, T\}$, $\{1,2,3,4,5,6\}$
  Tossing a coin once. Rolling a dice once.

- The event Space is the set/grouping of outcomes according to a chosen definition of an event.
  e.g. Rolling even no: $\{2,4,6\}$
  Flipping Heads: $\{H\}$
  The event space is structured as a σ-algebra for probability calculus to behave as intended.
  The grouping of the event space $E \subseteq \mathcal{P}(\Omega)$ is a subset of the power set of Ω.
  $\mathcal{F}$ must satisfy both the specified events and the σ-algebra conditions.
  Properties:

1) $\Omega \in \mathcal{F}$
2) $A \in \mathcal{F} \Rightarrow \overbrace{(\Omega \setminus A)}^{A^c} \in \mathcal{F}$
3) From 1 & 2, $\emptyset \in \mathcal{F}$
4) for $A_1, A_2, A_3, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$
   Closed under countable union
5) From 2 & 4, $\bigcup_{i=1}^{\infty} A_i^c \in \mathcal{F} \overset{\text{from 2}}{\Rightarrow} \left(\bigcup_{i=1}^{\infty} A_i^c\right)^c \in \mathcal{F}$
   De Morgan's Laws $\Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$
   $\mathcal{F}$ is also closed under countable intersection

examples of σ-algebras include

- $\mathcal{F}_1 = \{\emptyset, \Omega\}$
- $\mathcal{F}_2 = \mathcal{P}(\Omega)$
- $\mathcal{F}_3 = \{\emptyset, A, A^c, \Omega\}$

Generated σ-algebras

Collection of all possible events.

For any subset $\mathcal{T} \subseteq P(\Omega)$ of the powerset of the sample space, a unique minimal σ-algebra can be constructed from the sets in $\mathcal{T}$.
It is called the σ-algebra generated by $\mathcal{T}$ (denoted $\sigma(\mathcal{T})$).

↳ This is done by taking an arbitrary subset of $P(\Omega)$ ~~e.g. $\{\{1,2\},\{3,4,5,6\}\}$~~ and applying the σ-algebra axioms to it.

e.g. $\{\{1,2\}\}$

Step ① Add the complement of the set in $\mathcal{T}$
$\{1,2\}^c \to \{3,4,5,6\}$

~~① Add Ω~~
~~② $\emptyset$ follows from ①~~

Step ② $\Omega = \{1,2\} \cup \{3,4,5,6\}$

Step ③ $\Omega^c = \emptyset$ or $\{1,2\} \cap \{3,4,5,6\} = \emptyset$

$\mathcal{T} = \{\emptyset, \{1,2\}, \{3,4,5,6\}, \Omega\}$ is the smallest σ-algebra containing $\{1,2\}$.

e.g. $\{\underset{A}{\{1,2\}}, \underset{B}{\{3,4\}}, \underset{C}{\{5,6\}}\}$

→ A σ-algebra can be formed by any partition of Ω by including $\emptyset$ & all possible combinations of the union of the partitions.

$A \cup B \cup C \in \mathcal{T} = \Omega$
$A \cap B \cap C \in \mathcal{T} = \emptyset$ (or $\Omega^c = \emptyset$)
$A \cup B$, $A \cup C$, $B \cup C \in \mathcal{T}$
$(A \cup B)^c, (A \cup C)^c, (B \cup C)^c \in \mathcal{T}$

$\underline{\underline{C}}$ $\quad \underline{\underline{B}}$ $\quad \underline{\underline{A}}$

↳ Method of generating another smallest σ-algebra

~~σ-algebras generated by countable partitions~~ → ~~Grouping of elements such that every element~~

Borel σ-algebra (over $\mathbb{R}^n$)

A little bit of Topology

A topology on set X is a collection ~~of~~ $\mathcal{T}$ of subsets of X
$\mathcal{T} \subseteq P(X)$
s.t. the following axioms are satisfied:

1) $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$

2) Closed under arbitrary union
If $\{U_i\}_{i \in I}$ is any collection of sets in $\mathcal{T}$ then their union is also in $\mathcal{T}$:

$$\bigcup_{i \in I} U_i \in \mathcal{T}$$

3) Closed under finite intersection
If $U_1, U_2, U_3, \ldots, U_n \in \mathcal{T} \Rightarrow U_1 \cap U_2 \cap U_3 \cap \ldots \cap U_n \in \mathcal{T}$

$(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on X is called a topological space.
Sets in $\mathcal{T}$ are called open sets.

Open Sets

Metric Space def^n
for $U \subseteq (X,d)$ a set is open if for every point $x \in U$, $\exists r > 0$ s.t. the open ball $B(x,r) \subseteq U$.
$B(x,r) = \{y \in X \mid d(x,y) < r\}$ → Every point in U has a neighborhood contained in U.
$(0,1)$ is open because for any point $x \in (0,1)$ we can find a small interval around x that stays entirely within $(0,1)$. → Relies on def^n of limits & infinitely small...
$[0,1]$ is not open because the endpoints $0,1$ do not have neighborhoods that stay within the set.

~~Topology~~

Examples of Topologies

(a) Standard topology on $\mathbb{R}$.

$X = \mathbb{R}$

$\mathcal{T}$ is the collection of all open intervals (a,b) and the union of such intervals.

(b) Discrete Topology

$X = \{a, b, c\}$

$\mathcal{T} = \mathcal{P}(X)$

~~Note the striking similarity Topologies bear to σ-algebras. (although missing complementarity)~~

~~No σ-algebras are sets~~

(c) Trivial Topology

$x = \{a, b, c\}$

$\mathcal{T} = \{\emptyset, X\}$

Borel Sets and Borel σ-algebras.

Borel Set : They are NOT a subset of a topology.

A topology on set X is a collection of sets that satisfy the conditions for open sets.

A borel set is any set that can be formed from the open sets of a topological space through a combination of the following operations :

- Cantable union
- Cantable intersection
- Complementation

~~A borel set is like a σ-algebra but lacking the requirement for $\emptyset$ & X.~~

~~Is this true?~~

All open sets are borel sets but not all borel sets are open sets.

Book → A collection of Borel Sets make up the Borel σ-algebra. ← library

The Borel σ-algebra is the smallest σ-algebra containing $\mathcal{T}$

~~$\mathcal{B}(X)$~~ $\mathcal{B}(\mathcal{T})$ → $\mathcal{B} = \sigma$(open sets in $\mathbb{R}$)

→ The Borel σ-algebra is the smallest σ-algebra generated by the collection of open intervals/open sets in $\mathcal{T}$.

Borel Measure

A borel measure μ defined on the Borel σ-algebra $\mathcal{B}(X)$ of a topological space X assigns size to the ~~collect~~ sets in the Borel σ-algebra. ~~which itself~~

[~~Note: The Borel sets are a subset of a topological space. The collection of these subsets form a σ-algebra. The σ-algebra is a subset of the powerset of the topology on X.~~]

→ FALSE

The borel measure satisfy the following properties:

① $\mu : \mathcal{B}(X) \to \mathbb{R}$ (The domain of the measure is defined on $\mathcal{B}(X)$).

② $\forall A \in \mathcal{B}(X)$, $\mu(A) \geq 0$ (Non-negative measure)

③ $\mu(\emptyset) = 0$

④ σ-Additivity/Cantable additivity.

If $\{A_n\}_{n=1}^{\infty}$ is a sequence of disjoint Borel sets,

then $\mu\left(\dot{\bigcup}_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$

(disjoint union)

$X : (\Omega, F, P) \to (\mathbb{R}, \mathcal{Y})$

$\Omega \to \mathbb{R}$ ↘ Real Line

↓ distribution of outcomes

X maps outcomes of Ω to topological Space $\mathbb{R}$.

Now, that topological space forms a Borel Set $\mathcal{B}(\mathbb{R})$

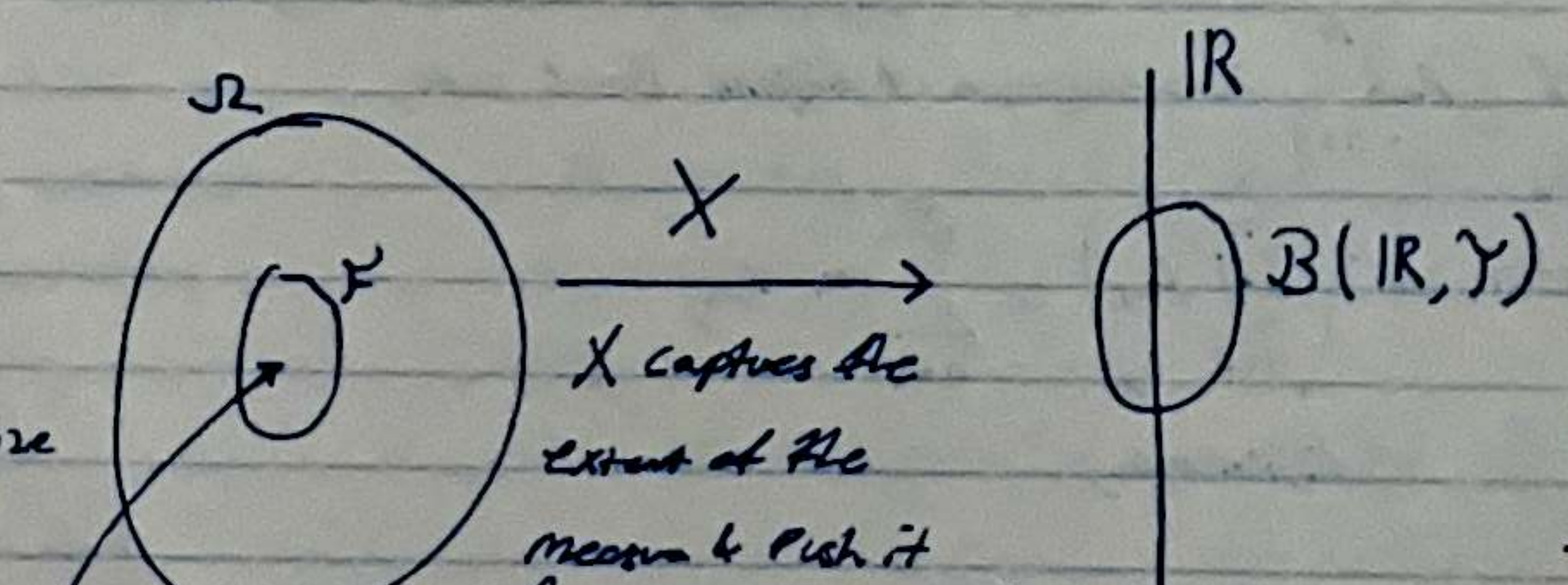
~~Borel Measure~~

$P : F \to \mathbb{R}$

Now P maps $\mathcal{F}$ onto $\mathbb{R}$

F is thus defined on $\mathcal{B}(\mathbb{R})$ too.

This is how events in F are linked to the real line through random variables.



X captures the extent of the measure & push it forward to the real line

This part is captured by a ~~[illegible]~~ probability distribution function.

measures size of sets

$\mathbb{P}$ measures the extent an outcome x lies in $\mathcal{F}$ relative to Ω.

X interprets the "extent" in a meaningful ~~and~~ and quantifiable way on a real line.

Probability Measure

$(\Omega, \mathcal{F}, \mathbb{P})$

$\mathbb{P} : \mathcal{F} \to [0, 1]$

- It is a measure defined from the event space σ-algebra to the interval $[0, 1]$.
- It is a special case of a borel measure, being normalized to $P(X) = 1$
- $\mathbb{P}$ is defined on the σ-algebra of events while μ (borel) is defined on borel σ-algebras.
- $\mathbb{P}$ is used to assign probabilities to events.

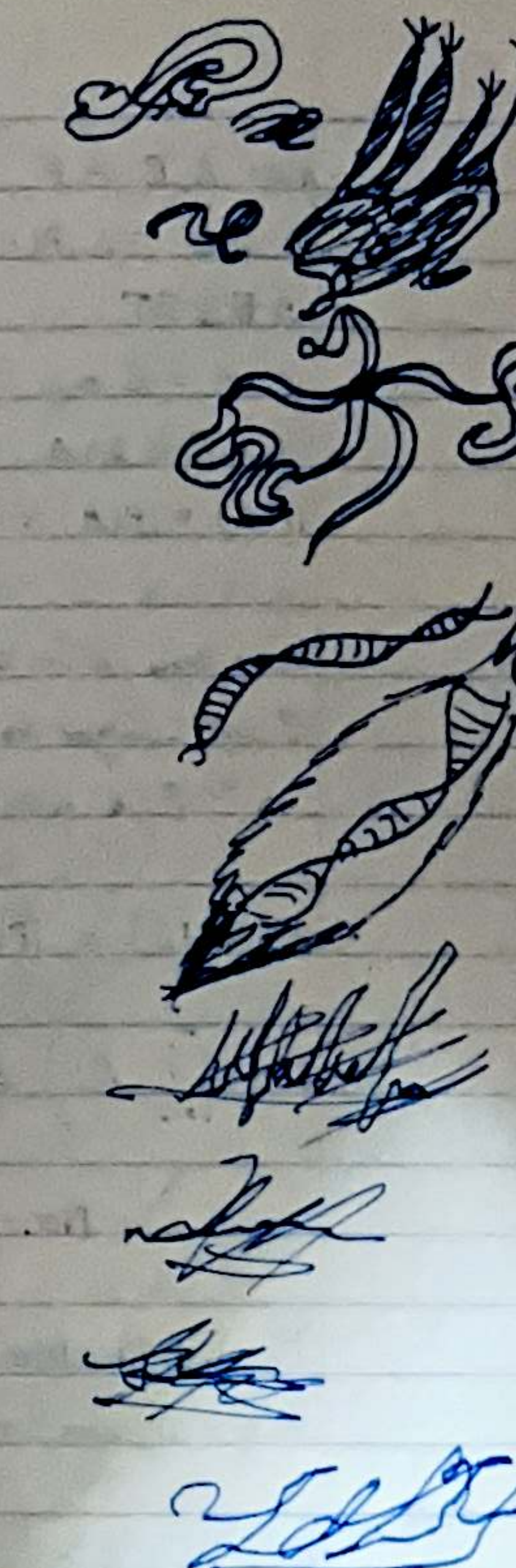
Desirable Properties of $\mathbb{P}$.

1) $\mathcal{F} \subseteq \mathcal{P}(\Omega)$

$\exists\, \mathbb{P} : \mathcal{F} \to [0, 1]$

s.t. : $\mathbb{P}(\Omega) = 1$

$\mathbb{P}$ is σ-additive $\to \mathbb{P}\left(\dot{\bigcup}_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$



Calculation Properties

- $\mathbb{P}\left(\dot{\bigcup}_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$

  ↳ The measure probability of a finite union is the sum of their individual probabilities.

- $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

  $\underbrace{\mathbb{P}(A \cup A^c)}_{} = \mathbb{P}(A) + \mathbb{P}(A^c)$

  $\mathbb{P}(\Omega) = 1 = \mathbb{P}(A) + \mathbb{P}(A^c)$

  $\Rightarrow \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

- $\mathbb{P}(\emptyset) = 0$

  $\mathbb{P}(\emptyset) = 1 - \mathbb{P}(\Omega)$

- For $A, B \in \mathcal{F}$

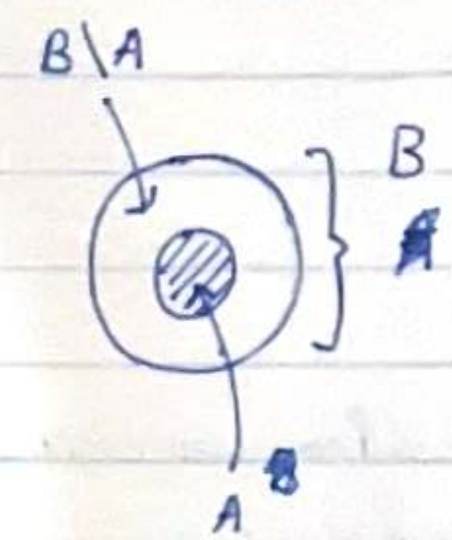
  $A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

  ~~$A \cup B \setminus A$~~

  $B \setminus A = B$ only

  $B = A \cup B \setminus A$

  $\mathbb{P}(B) = \mathbb{P}(A) + \underbrace{\mathbb{P}(B \setminus A)}_{\geq 0}$



For any A or B,

B will always be greater than A

$\Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

- $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

  ↳ Non-disjoint Union in these cases.

Probability Measures help assign weights to the elements (event set collection) of a σ-algebra so as to characterize likelihood of events.

$$\boxed{\text{Define } \mathbb{P} : \mathcal{F} \to [0,1] \begin{cases} \mathbb{P}(\emptyset) = 0 \\ \mathbb{P}(A) = 1 \end{cases}}$$

↳ IP must always satisfy the new conditions as well as the past properties.

Examples

$\Omega = \{1, 2\}$

$\mathcal{F}_0 = \{\emptyset, \Omega\}$ $\quad \mathcal{F}_1 = \mathcal{P}(\Omega) = \{\{\emptyset\}, \{\Omega\}, \{1\}, \{2\}, \{1 \cup 2\}\}$

$\mathbb{P}$ is a probability measure on $\mathcal{F}_0$ as $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$

$\boxed{\text{but not on } \mathcal{F}_1}$ as $\underbrace{\mathbb{P}(\{1 \cup 2\})}_{\mathbb{P}(\Omega) = 1} = \underbrace{\mathbb{P}(\{1\})}_{1} + \underbrace{\mathbb{P}(\{2\})}_{1} = 2$ ↯

Laplace definition of Probability

Let $\mathcal{F} = \mathcal{P}(\Omega)$

$\forall A \subseteq \Omega \quad \mathbb{P}(A) \overset{\text{def}}{=} \frac{|A|}{|\Omega|}$ (Number of events divided by total outcomes)

Size

$\mathbb{P}$ is a probability measure on $\mathcal{F}$

$\mathbb{P}(\Omega) = \frac{|\Omega|}{|\Omega|} = 1$

$\mathbb{P}$ is σ-additive (finite additive)

$\mathbb{P}\left(\dot{\bigcup}_{i=1}^{n} A_i\right) = \frac{1}{|\Omega|} \left|\dot{\bigcup}_{i=1}^{n} A_i\right| = \frac{1}{|\Omega|} \sum_{i=1}^{n} |A_i| = \sum_{i=1}^{n} (A_i)$

Dirac Measure

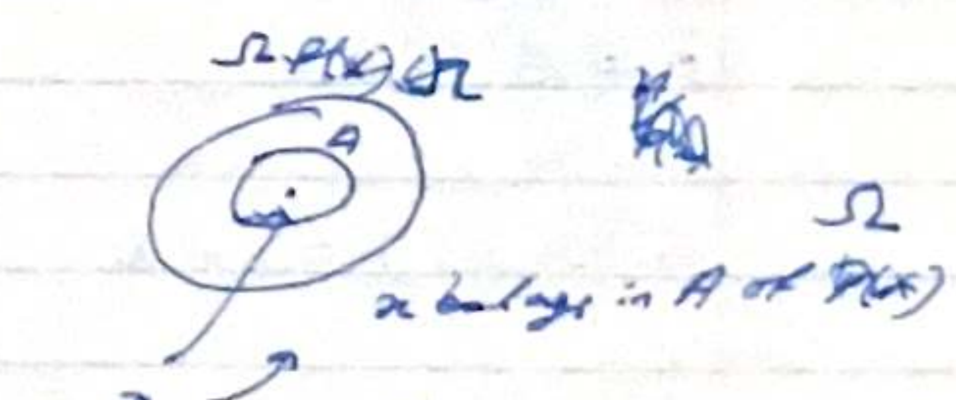
Let $\mathcal{F}$ be a σ algebra

Let $x \in \Omega$

$\mathbb{P}_x(A) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$

is a probability measure $\delta_x((A))$

→ Categorize whether the a is in the event or not



→ A measure quantifies the likelihood of an outcome belonging to the subset of event relative to the entire sample space (or powerset of all other possible events)

Discrete Measure (Power Set)

$\mathcal{F} = \mathcal{P}(\Omega)$

$p : \Omega \to [0,1]$ s.t. : $\sum_{x \in \Omega} p(x) = 1$

$\mathbb{P}(A) \overset{\text{def}}{=} \sum_{x \in A} p(x)$ for all $A \subseteq \Omega$

↳ Probability measure on power set

Example

Event space being all possible combination of events

$\Omega = \{1,2,3\}$ $\mathcal{F} = \mathcal{P}(\Omega)$

$p : \Omega \to [0,1] : p(1) = 0.4, \; p(2) = 0.4, \; p(3) = 0.2$

Probability measure assesses likelihood of outcomes

$$\sum_{x \in \Omega} p(x) = p(1) + p(2) + p(3) = 1$$

$$\mathbb{P}(A) = \sum_{x \in A} p(x) \quad \text{for all } A \subseteq \Omega$$

$$\mathbb{P}(\{1,2\}) = p(1) + p(2) = 0.8$$

Event Space groups outcome as desired

σ- algebra ensures that combining individual outcome probability to give event probability follows rule of probability.

Probability Measure over an uncountable set.

$\Omega = [0,1]$
$\mathcal{F} = \mathcal{B}([0,1])$

$\lambda([a,b]) = b - a \qquad [a,b] \subseteq [0,1]$

$\lambda(\Omega) = \lambda([0,1]) = 1$

~~Absolute~~

Number between 0 & 1 is chosen at random, uniformly.

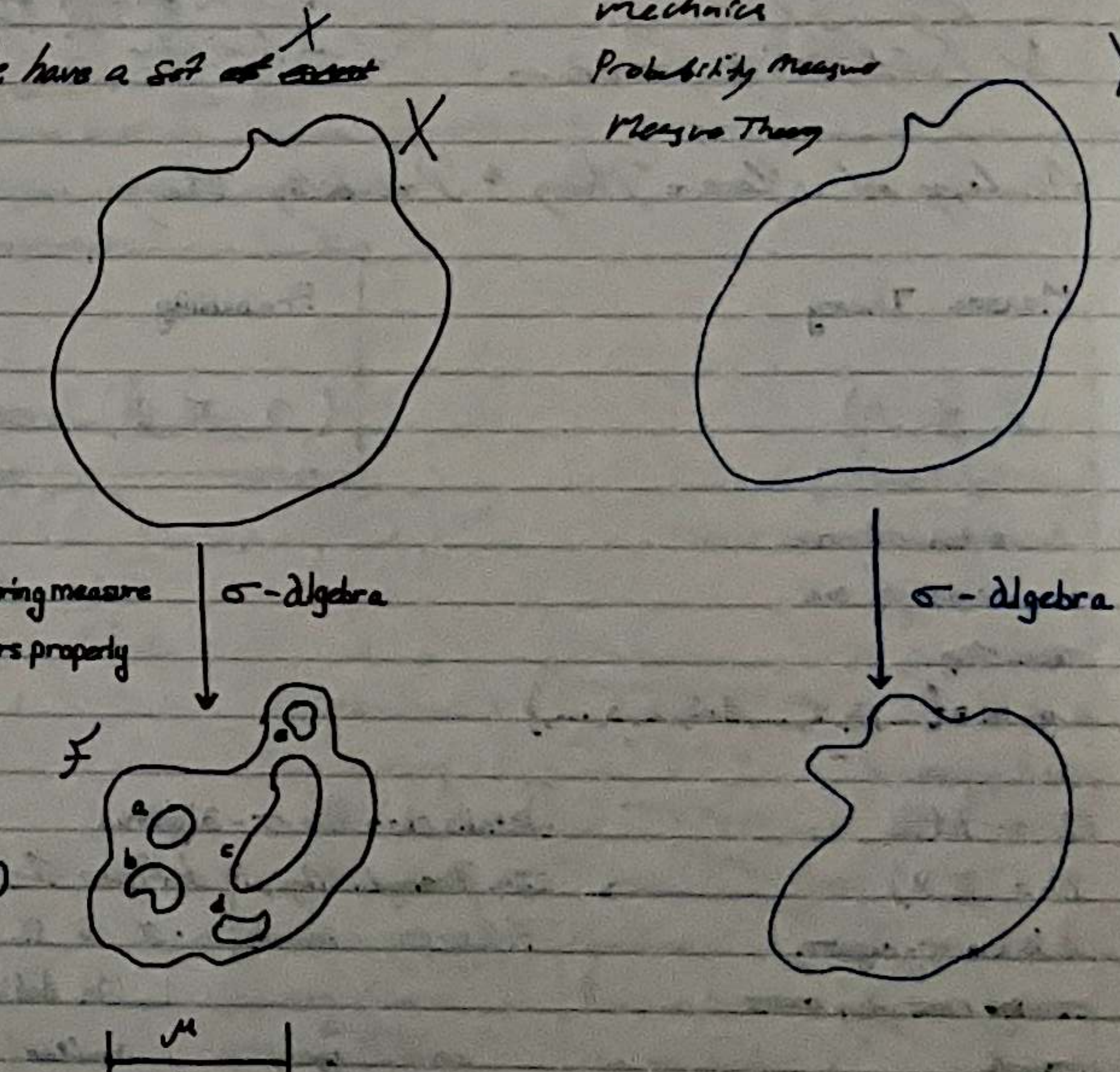
Interpretation of that probability space.
$([0,1], \mathcal{B}([0,1]), \lambda)$

~~Random Variable~~

Larger Picture.

We have a set ~~of event~~ X

Expectation value
statistical mechanics
Probability measure
Measure Theory



Ensuring measure occurs properly

Contains
∅ & Ω
$\mathcal{F} \subseteq \mathcal{P}(X)$

Assigns size to each of these Subsets.

Have a list of info how big each Subset is

e.g. a=3 c=10 e=3
b=3 d=2

Analogy of Measure Theory & Probability Theory.

Measure Theory

$(X, \mathcal{B}, \mu)$

Probability

$(\Omega, \mathcal{F}, \mathbb{P})$ $\mathbb{P}(\Omega) = 1$

X is the universe from which we are measuring.
e.g. $\mathbb{R} = \{\ldots, 0, 1, \ldots, 1.5, 2, 3, \ldots,\}$

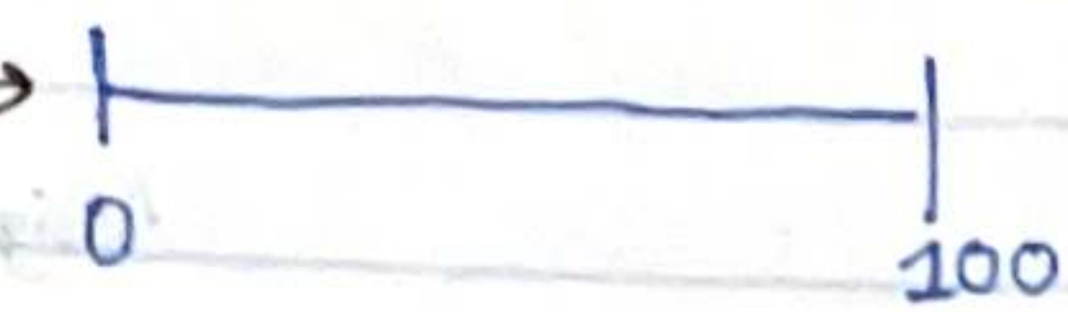
~~$\mathcal{B} = \mathcal{B}(\mathbb{R}$,~~
$\mathcal{B} = (\mathbb{R}, \mathcal{B})$ ⟶ Details about the σ-algebra. In Probability, it defines the event.

$\mathcal{B}$ is the σ-algebra ensuring measuring occurs properly.
It contains subsets formed by the elements of $\mathbb{R}$.
It is seeded by $\mathbb{R}$
e.g. $\{\emptyset, \mathbb{R}, \{1\}, \{1\}^c\}$
↳ This is where the size of subsets are assessed.

$\mu : \mathcal{B} \to \mathbb{R}$
μ is a measure function assessing the size of sets.

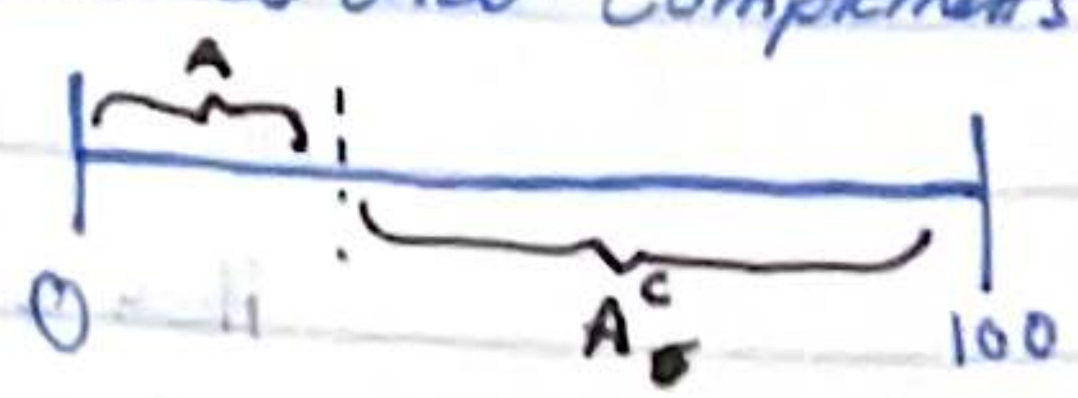
Philosophy : Axioms ⇒ • $\emptyset$ & Ω included
↳ We define the largest and smallest elements when measuring

Suppose we get a rod. We mark the start & end. →



• Closed under Complements.

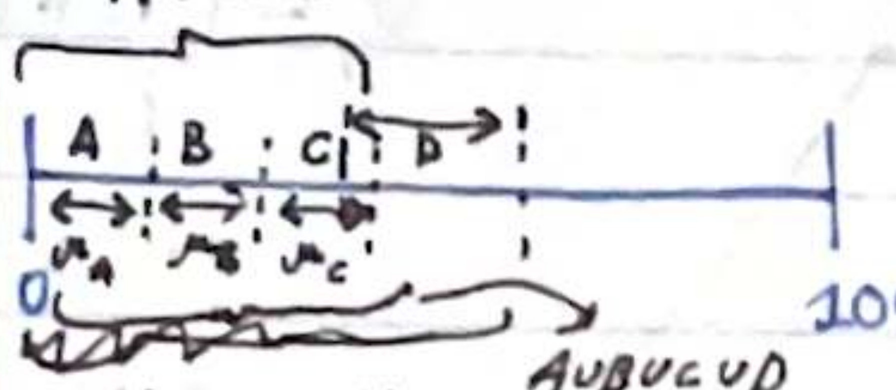
However I divide it, there is always one part I choose, & another that I reject (The complement)
A division* splits into choosing & rejecting



* Dividing depends upon choosing A or B or C...

Closed under Union

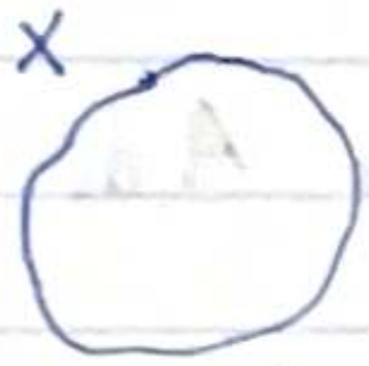
AUBUC



It's not that deep I suppose.
A bigger part can be made from smaller parts.

Measurable functions

X



σ-algebra

μ

Size of Subsets (Splittings)

There are 10 red fruits
2 yellow fruits
1 green fruit
0 blue fruit
5 orange fruits
⋮

· Sets of fruits.

· Dictates how the splitting is done.

[e.g. Segregate based on colour]

↳ must take in account complement, union and stuff like that (or use λ)

→ The random variable's definition is intricately tied to how the splitting is done (exchange)
If we segregate based on color in σ, then let r.v. be the color of fruit.

$f: X \to \mathbb{R}$
$f^{-1}(X) \subseteq \sigma$ algebra

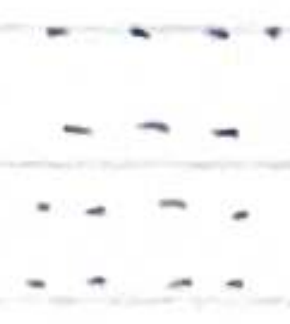
Measurable function transfers the σ-algebra structure to this one

Information gets interpreted in terms of $\mathbb{R}$.

Probability mass distribution.
PMF

$\mathbb{R}$

Borel σ-algebra



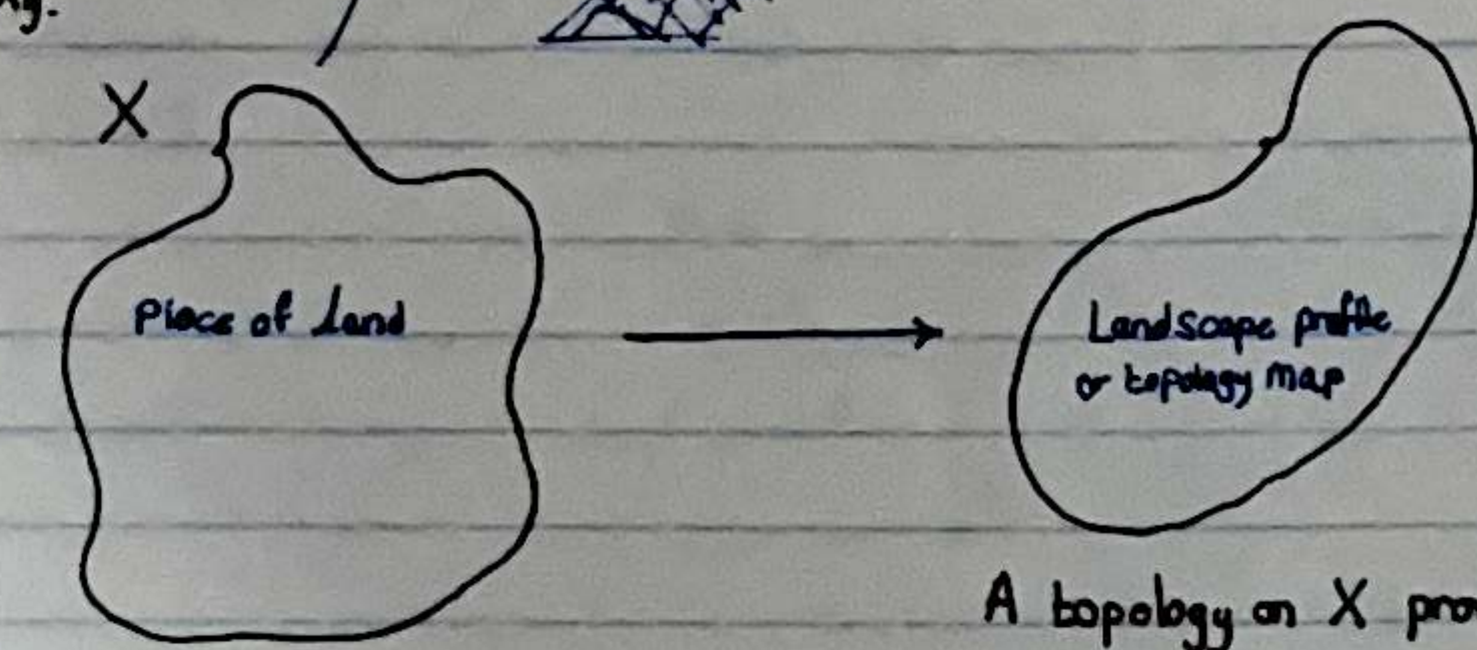
Bring it into being.

Collection of Stuff

Philosophy

Phenomenology.
↳ Measurements is inherently observer dependant. Measurements is inseparable from Some choice or selection.

X



Some Set X being a collection of Stuff

$(X, \mathcal{T})$

$\mathcal{T}$ is a collection of Subsets of X Such that Smooth "connections" between data points is meaningful.

A topology on X provides the "Shape" ~~and~~ of the data being the "closeness" of points in the set.

Can you measure something w/o choosing to measure it?

(X, σ)

σ - Algebra (measurable space)

Question, why do ants specifically define a σ-algebra.

↳ What do we want to measure is a good question.

Example: Height of each region of the map or size of rock formation *

This is done by the measure function μ.

Implicitly Choosing the parts we will measure.

[Partitioning Land into parts which can be measured.

[Size of rocks
Height of land
Dryness of land

[A σ-algebra (allows us to measure the "volume"/"importance" of each of these Shapes on the Set.)

↳ Split the land in such a way that it can be measured.

How to measure
↳ What even is measuring stuff?
↳ Philosophy

μ : Measure Space
(X, σ, μ)
↳ Info about the size of each land.

· Axioms : $\varnothing$ & Ω included
or ∞

0 — 100 Smallest & largest element included.

$\{ \varnothing \mid \Omega \}$

How do I interpret complement & union in terms of measuring land height

Height of villages above sea level : $\varnothing$

+

Measure & Measurable Space = Measure Space

μ . (X, Σ) (X, Σ, μ)

$\left\{ \begin{array}{l} \varnothing, \Omega \\ \text{Countable union closure} \\ \text{Complement closure} \end{array} \right]$

≥ 0

Countable additive ✓

Elements of a measurable Space are measurable Subsets

Why do the axioms make them measurable?

What can we measure?

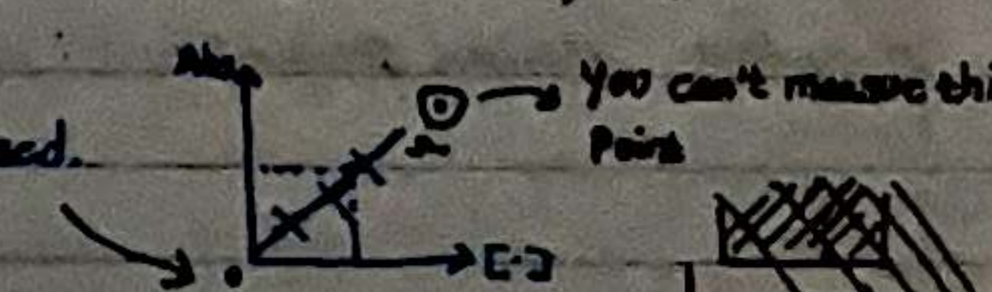
* A measurable Space is a set equipped with a σ-algebra Structure.

(X, Σ)
↳ Collection of subsets of X.

· In our case, a Subset would correspond to choosing a region of the landscape profile.

Axioms : There is an empty region ($\varnothing$) and the whole region (X or Ω) as part of the measurement Structure.

① The measurement System is closed.



→ You can't measure this Point

② Closed under complementarity.
↳ When we choose a subset A of landscape, we implicitly state that it is not A^c (relative to universal set Ω)

→ Can't measure this point too.

Ruler

You can't measure something without choosing to measure it.

Same as choosing ones.

③ Closed under union
↳ Choosing pieces of land A & B together means as a single choice of larger piece of land C

$C = A \cup B$

Measure Function

Optional !!

Piece of Land X → (Shape of Land or "closeness" of points $(X, \mathcal{T})$) → Split Landscape into parts to measure (X, Σ) → Measure information (X, Σ, μ) → Measure What

$(X, \mathcal{T})$ → The contour or topology

(Closeness) in terms of what? → Proximity.

How similar they are to each other → related → How much they belong to the same open set...

Not related based on a single property but related as a whole.

If 2 points lie within the many of the same open sets, they are considered close.

Whatever we want

Laplace Classical probability : $\frac{|A_i|}{|\Omega|}$ associated with region/measure/open set i

Counting Measure $\mu(A_i)$ = nº of elements in A_i

Dirac Measure : $\delta_a(A_i) = \chi_{A_i}(a)$ $= 1$ if $a \in A_i$, 0 if $a \notin A_i$

→ indicator function of A

Borel Measure → Gaussian Measure

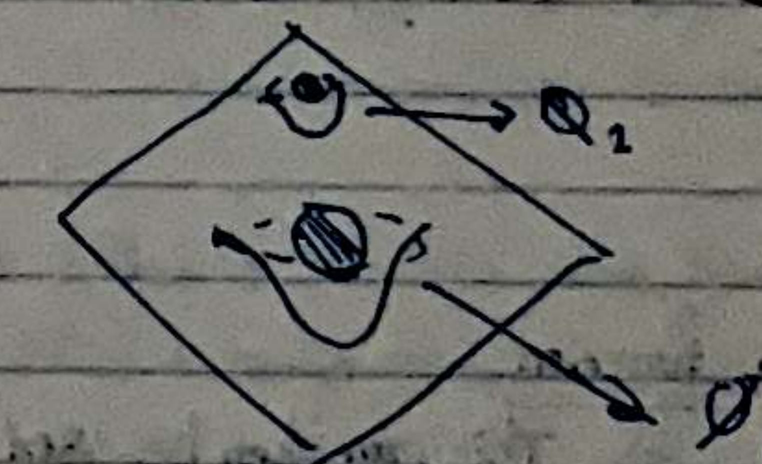
Liouville Measure → Liouville Theorem in classical Mechanics

→ Natural Volume form on a Symplectic manifold.

Gibbs Measure → Canonical Ensemble

Physics → Special distribution of Mass

↳ Gravitational Potential → Makes so much fucking sense!!!



Properties of Measure

① Monotonicity → output consistently increases or decreases as input increases

A less rigid form of linear response → Required proportionality.

E_1 & E_2 are measurable sets with $E_1 \subseteq E_2$ then $\mu(E_1) \leq \mu(E_2)$

② All the properties we previously saw in connection with probability.

Countable subadditivity
Continuity from below
Continuity from above
Complete measure
Additivity

{ Many of the laws of probability is simply measuring probability.
The question remains, what is probability? }

Real Measurable function

$\boxed{f : X \to \mathbb{R}}$

So far, μ gave us the following :

Subset (Region)	"Size" of a measured property
⋮	⋮

Intuitively, the measurable function f maps the piece of land to $\mathbb{R}$ while ensuring measured properties are preserved.

Why would we want to do that ?

It also transfers the measure μ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and the new push forward measure is denoted $f_{\#}\mu$.

It turns the measure into a function.

This can then be used for further analysis

This is then how Clustering is used to derive Gaussian Distribution equation

The measure is interpreted as a function/rule for the Sizes of the Subsets. This is the purpose of the measurable function.

* It forms a function from the table we obtained.

That function can then be used for mathematical analysis.

$\mathbb{R}$ (Real Analysis)

$\mathbb{C}$ (Complex Analysis)

f maps $(X, \Sigma) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$\boxed{\forall B \in \mathcal{B}(\mathbb{R}) \quad f^{-1}(B) \in \Sigma}$ $\int_X f \, d\mu$ is well defined

e.g. ~~Gaussian Distribution & Gaussian Measure~~

We always integrate measurable functions w.r.t a measure.

Integration Measures and Measurable functions

Lebesgue Integration w.r.t measure μ

$\boxed{\int_X f \, d\mu}$

$f : X \to \mathbb{R}$ is a measurable function

X is the domain of integration

μ is measure defined on measurable space $(X, \mathcal{B})$

$d\mu$ indicates integration performed w.r.t μ

?? ~~For every subset A in X, f converts that value into a $\mathbb{R}$ quantity while accounting for weighing based on μ.~~

size

μ determines how the size of the subsets of X contribute to the integral.

$f : X \to \mathbb{R}$. $f : a \mapsto x$ where $a \in X$ and $x \in \mathbb{R}$.

$\mu = \lambda(x)$ Lebesgue Measure

Lebesgue Integral $= \int_X f(x) \, d\lambda(x)$

~~f is a measurable function~~ Lebesgue

↳ We are integrating over the domain X. The range of $f : X \to \mathbb{R}$ is $f(x) \in \mathbb{R}$. We then weigh the range based on the size of the subsets e.g. A_i in the domain using the measure λ.

X

f is a measurable function.

Lebesgue's theory define integration for a class of functions called Measurable functions.

(X, Σ, μ) $f : X \to \mathbb{R}$

A function f on X is measurable if the preimage of every interval of the form (t, ∞) is in X :

$\{x \mid f(x) > t\} \in \Sigma \ \forall t \in \mathbb{R}$

Measure Theory → $f(x)$ is same as X → r.v

↖ Probability

Let r.v X be sum of dice $\boxed{X = d_1 + d_2}$

~~f~~ $\boxed{f : d_1 + d_2 \mapsto x \in \mathbb{R}}$ $\boxed{f(x) = d_1 + d_2}$

$(\Omega, \mathcal{F}, \mathbb{P})$

~~ex~~

Probability

It makes sense that if we have a topological landscape and we measured ~~an~~ ~~average~~ a quantity for each region, and we converted that variation in quantity to a function.

The "average" value of that function would correspond to the average value of that quantity over the topological landscape.

In probability spaces, the measure $\mathbb{P}$ would give the probability associated with each region

e.g. $\mathbb{P}(A_i)$ for region $A_i = \frac{|A_i|}{|\Omega|}$

The function X which interprets a subset as a value on $\mathbb{R}$ is the representation of that subset. X is called a random variable.

We then weigh the (interpretations of the subsets) with the probability measures

X_{A_i} or $f(\omega^{A_i})$ — $\mathbb{P}(A_i)$

This gives the average value of the interpretation of the subsets.

This corresponds to the expected value ~~E~~ ~~E~~

$\cancel{\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P}}$

$\int_{\mathbb{R}} t \, d X_{\#}\mathbb{P}$ → t is the subset converted to $\mathbb{R}$

Domain ← $\mathbb{R}$ in $\mathbb{R}$ too, not only range

$$\boxed{\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P}}$$ → The expectation of X.

random variable.

Note: X must be properly defined...

Can't define random variables in any stupid ways we want

It must match the σ-algebra $\mathcal{F}$

$X^{-1}(A)$

$\boxed{X^{-1}(x) \in \mathcal{F}}$

~~Push forward Measure & Integrating on $\mathbb{R}$~~

Variance in Probability Space & its equivalence in Measure Spaces.

↳ Just L^2-norm of $(\mathbb{E}[X] - X)$.

~~To Summarize real measurable functions & Lebesgue integration.~~

~~Why do we even define a measurable function and what are its properties.~~

- ~~It allows for mathematical analysis as we are dealing with functions instead of a pair (subset, quantity)~~
- ~~the pair was generated by a measure. The measurable function turns those into a rule/function.~~
- ~~It also allows for quantification of~~

~~Generating a Gaussian Random Variable~~

$\boxed{\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P}} = \boxed{\int_{\mathbb{R}} x \, d X_{\#}\mathbb{P}(x)}$ → Working in $\mathbb{R}$ only.

~~Both is the same thing~~

Integrating over the subsets in the measure space w.r.t the measure $\mathbb{P}$.

Integrating over $\mathbb{R}$ w.r.t the pushforward measure $X_{\#}\mathbb{P}$ induced by measurable function X.

↳ The pre-image $X^{-1}_{\#\mathbb{P}}(x)$ ~~as~~ of the induced measure $X_{\#}\mathbb{P}$ must lie ~~in~~ in the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

The pre-image $X^{-1}(A_i)$ of the measurable function X must be a subset present in the σ-algebra $\mathcal{F}$.

Boolean algebras & Sigma algebras

Σ-algebras

Set of subsets of X

A σ-algebra over a set X is a subset $\Sigma \subseteq \mathcal{P}(X)$ satisfying:

1) For i countable, $U_i \in \Sigma \Rightarrow \bigcup_{i=1}^{\infty} \in \Sigma$ (countable) $\begin{cases} I \text{ countable set } (0 \leq |I| \leq \omega) \\ i \in I \end{cases}$

2) $U \in \Sigma \Rightarrow X \setminus U \in \Sigma$

A measurable space is a pair (X, Σ) where X is some set and Σ is a σ-algebra over X.
In such situations, a subset $S \subseteq X$ is called measurable iff $S \in \Sigma$

Note: It follows from 2 that $\emptyset \in \Sigma$, $X \in \Sigma$
Also, Σ is closed under countable intersection.

Fix a measurable space (X, Σ). A measure on (X, Σ) is a function $\mu : \Sigma \to [0, \infty]$ s.t

1) $\mu(\emptyset) = 0$

2) $i \in I$ countable, $U_i \in \Sigma$

$U_i \cap U_j = \emptyset$ for $i \neq j$
↳ Disjoint sets

then $\mu\left(\bigcup_{i \in I} U_i\right) = \sum_{i \in I} \mu(U_i)$

A measure space is a triple (X, Σ, μ) where (X, Σ) is a measurable space and μ is a measure on (X, Σ)

$\boxed{\text{USRA}}$ → Ask Bruce & other Teachers for paid pos.

Boolean Algebras

Partition

Equivalence class cause decomposition.

Review of Order Theory → Previously reviewed in Binary relations

Posets

A poset is a set equipped with a partial order. → A binary relation satisfying certain properties
↳ A binary relation on set P is a rule or condition determining whether two elements of P are related. It is a subset of $P \times P$.

$(P, \leq)$
↳ A set ↳ Binary relation on P satisfying partial order properties.

e.g. relation $\in$; $A \in B$ can be true or false.
It is a relation that is either true or false, why or is it called binary because 2 elements are involved? ↳ It is called binary because 2 elements involved.

Binary Relations may have some or none of the following properties.

1) Reflexive, $\forall a \in P$ $a \leq a$
2) Symmetric, $a \leq b \Rightarrow b \leq a$
3) Antisymmetric, if $a \leq b$ and $b \leq a$ then $a = b$
4) Transitive, if $a \leq b$ and $b \leq c$ then $a \leq c$
↳ These can represent any relation is greater than, is a friend of, is better than, is divisible by, ...

[The true/false value is a property of all relations. ↳ Relations are fundamental mathematical relationships ↳ equivalence relations ?? $x \sim y$]

Partial order, $\leq$ is a special type of binary relation satisfying the properties of: Reflexive, Antisymmetric and Transitive. → Symmetric not necessary

Reflexive: every element is related to itself

Antisymmetric: No two distinct element can relate to each other in same way.

Transitive: Relation chains consistently across elements.

For set P, $(P, \leq)$ meaningful relation.

- Elements are members of the set P
- Two elements $a, b \in P$ are comparable if either $a \leq b$ or $b \leq a$
- Two elements $a, b \in P$ are incomparable if neither $a \leq b$ nor $b \leq a$

· A total order on P is a poset where all elements of P are comparable.
· A partial order allow for incomparable elements.

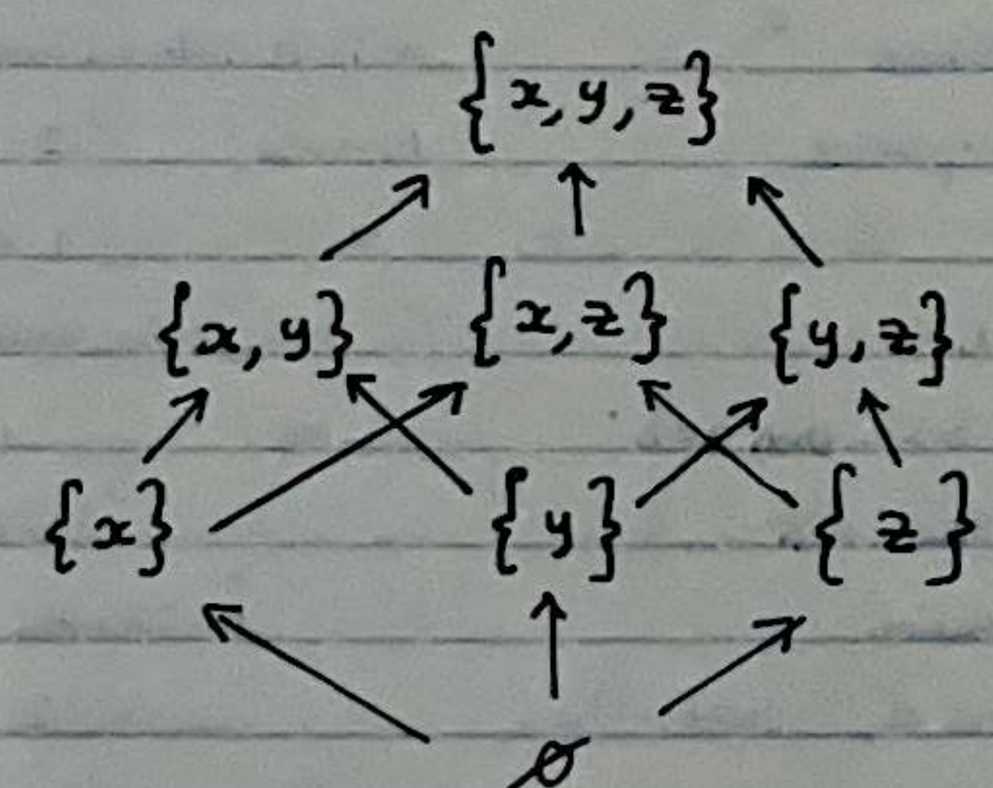
Examples of Posets

$(P, \leq)$ where $P = \mathcal{P}(\{x,y,z\})$ and $\leq$ is $\subseteq$ Subset relation.

Easy to show that $\subseteq$ is reflexive, Antisymmetric and Transitive.

① $P = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, \{x,y,z\}\}$

$R = \subseteq$
$(\leq)$



② $P = \{1,2,4,8,16\}$
$\leq$ = "divides"

③ $P = \mathbb{R}$
$\leq$ = usual "less than"

Key Elements in a poset.

When we say minimum/maximum, we mean in terms of order and not "size".

- Minimal Element   ~~least~~
- Maximal Element
- Least Element
- Greatest Element
- Least Upper Bound/Suprema
- Greatest Lower Bound/Infima

Dual

Do category of posets admit co-posets like posets less than form poset greater than?

- Suppose we have a poset P and S is a subset of the poset P.
  $x \in S$ minimal s.t $\nexists y \in S$ $y < x$.
  There is no element which x is "less than" or there is no element which x is "greater than".
  This is not the same as least element.
  There can be multiple minimal elements in a set but the minimal is not necessarily the least.

Tuning detect signal
↳ Resonance frequency, $Reactance_L = Reactance_C$
↳ $I \to \infty$
Maximum Amplitude oscillation.

Filter Parallel L-C Circuit.

What to reject → Cue depression.
↳ Fear of death
↳ Constant Growth

Maximal Element

$x \in S$ is ~~minimal~~ maximal in a Subset S of Poset P if there is no element $y \in S$ s.t $x < y$.

$\boxed{\nexists y \in S \text{ s.t } x < y \Rightarrow x \text{ is maximal.}}$

Minimal & Maximal Elements that have no "smaller" or "larger" elements within a subset, but they don't need to be comparable with all other elements.

e.g. Under $\leq$ for $S = \{2, 3\}$

Minimal = 2, Maximal = 3 } In this case, they also correspond to least and greatest.

e.g. $\{2,3\}$ under divisibility

Minimal elements : 2 & 3 (neither is divisible by the other)
Maximal elements: 2 & 3 (neither divides the other)

Least and Greatest Elements refer to absolutely the smallest or largest element in a subset S of a poset P. They are comparable to all elements in S and they are unique if they exist.

Least Element, $x \in S$ is least element of S if $x \leq y \ \forall y \in S$

Greatest Element, $x \in S$ is the greatest element of S if $y \leq x \ \forall y \in S$.

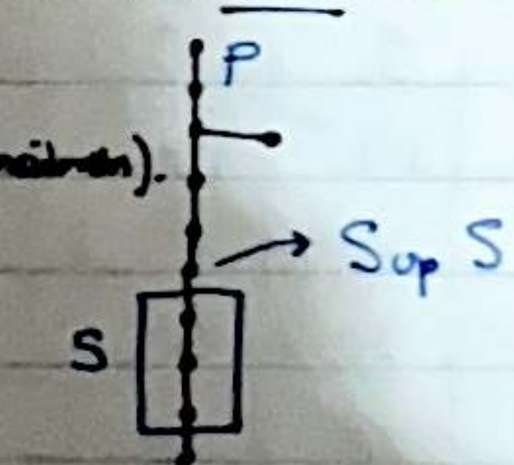
e.g. $\{1,2,3\}$ under $\leq$
Least : 1
Greatest : 3

$\{2,3,5,6\}$ under divisibility
Least : 2 (divides all other elements)
Greatest : No Greatest Element.
↳ No single number is divisible by all other.

Supremum & Infimum
↙ Sup ↘ inf

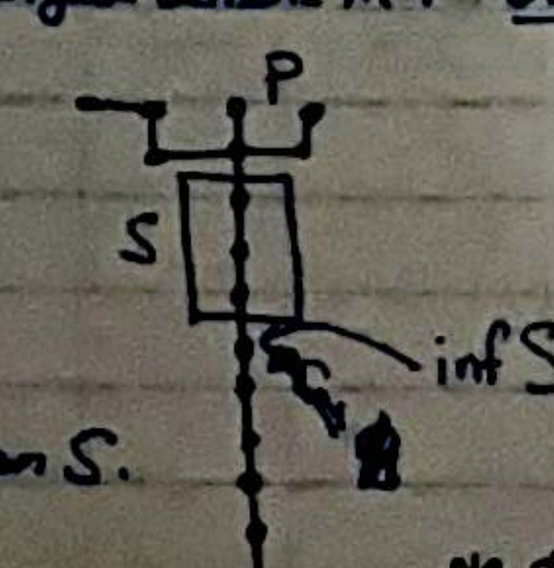
These describe bounds relative to the whole poset, not just the Subset.
We say the Sup of a Subset, not the whole Poset.

Supremum (Join, Least Upper Bound)
The Supremum of a Subset $S \subseteq P$ is the Smallest element in P that is greater than or equal to all elements of S.
↳ The Supremum is global to P (takes whole Poset in consideration).
It acts as a bound to Subset S.



Conditions : Greater or equal to all of S
AND the Smallest element in P that satisfies the above condition.
→ if Sup is in S, it is the greatest element
not in S → Act as a bound
it may or may not exist if poset lacks sufficient elements.

Infimum (Meet, Greatest Lower Bound)
The infimum of a Subset $S \subseteq P$ in a poset P is the largest element in P that is less than or equal to all elements in S.
inf $S \in S \Rightarrow$ least element
inf $S \notin S \Rightarrow$ Bound



e.g. $[0,1] \subseteq \mathbb{R}$ under $\leq$ — largest in P that is less than S.
Sup = 1
inf = 0

$\{3,5,7\}$ under divisibility.
↳ Sup & inf do not exist
↓
No single number is divisible or divides all 3.

Probability

- The sample space Ω is the set of all possible outcomes.
  e.g. $\Omega = \{H, T\}$, $\{1,2,3,4,5,6\}$, $\{R, B, G, Y, O, P\}$
  single ~~coin flip~~ coin flip — single dice toss

- * An event specifies the outcome we are interested in. It is a subset of the sample space.
  e.g. $\{4\}$ rolling 4
  $\{2,4,6\}$ rolling an even number
  The elements of the event space all satisfy the event specified and none other.

- A σ-algebra ($\mathcal{F}$) is a collection of subsets of the sample space Ω that satisfy the following properties.
  $\Omega \in \mathcal{F}$
  $\emptyset \in \mathcal{F}$
  if $A \in \mathcal{F}$ then $\Omega \setminus A \in \mathcal{F}$
  if $A_1, A_2, A_3, \ldots \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

  ~~You don't need to think to live.~~
  ~~You don't need to be smart to live.~~
  ~~You don't need to solve problems to live.~~
  ~~Don't think too much.~~

  These axioms allow for a measure space $(\Omega, \mathcal{F}, \mu)$ to be defined. Those subsets which are measurable ~~can be~~ are in the σ-algebra and can be assigned probabilities
  * An event is a subset that: 1) satisfy the event specified
  2) belong to $\mathcal{F}$
  This ensure that probability can be defined for these events.
  → Events are elements of the σ-Algebra and the latter determines which subset of Ω are valid events.

- A probability space $(\Omega, \mathcal{F}, P)$ is a measure that ensures all elements of $\mathcal{F}$ sum up to 1.
  It obeys the axioms of the σ-algebra as well as the following:
  $P(\emptyset) = 0$
  $P(\Omega) = 1$
  $P(A) \geq 0$
  $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

These ensure that probability is calculated as traditionally done.

1. $\emptyset, \Omega$
2. Closed under Complementarity
3. Closed under (finite) Union

σ-algebra axioms

↳ This is very fucking arbitrary, It depends on how we define the measure

↳ Laplace Measure

↳ Dirac Measure

↳ Lebesgue Measure

For intuition of measure, need more details into σ-algebras

Like selecting the height of a rock, we select an event...

More details into σ- algebras

σ- algebras are a collection of subsets of set X

or σ- algebras are each subsets of the power set of X.

Defn: $\mathcal{A} \subseteq P(X)$ s.t. $\emptyset, X \in \mathcal{A}$

if $A \in \mathcal{A}$ then $X \setminus A \in \mathcal{A}$

if $A_i \in \mathcal{A}$, $i \in \mathbb{N}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

then $\mathcal{A}$ is called a σ- algebra.

$A \in \mathcal{A}$ is then a measurable set

Sigma algebra

We want to find the smallest $\mathcal{A}$ on X \$ given we have a lot of $\mathcal{A}$'s on X.

$\bigcap_{i \in \text{Index}} \mathcal{A}_i$ is also a σ- algebra $\longrightarrow$ Intersection of 2 σ- algebras will give a new smaller one.

⋮

Can keep this going for however many indices we want and we will keep getting smaller & smaller.

Suppose we are interested in a particular group of outcomes in a Sample space ~~(a family of~~ (a family of subsets in the power set of X), $\mathcal{M}$.

e.g. $\{\{1,2\},\{3,4\},\{5,6\}\}$ or $\{\{1,2,3\},\{4,5,6\}\}$

For that $\mathcal{M} \subseteq P(X)$, there is a smallest σ- algebra that contains that subset $\mathcal{M}$.

As we saw earlier, intersection yields a smaller σ- algebra,

$$\bigcap_{\substack{\mathcal{A} \subseteq \mathcal{M} \\ \mathcal{A} \text{ is a } \sigma\text{-algebra.}}} \mathcal{A} \quad \rightarrow$$

More details into σ-algebras

$(\Omega, \mathcal{F}, P)$

Ω → Sample Space

$\mathcal{F}$ → Event Space

P → Probability Measure

Let $\Omega \neq 0$ be a set. Let $\mathcal{P}(\Omega)$ be its power set.
Then $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is a σ-algebra if it satisfies

- $\Omega \in \mathcal{F}$
- $\emptyset \in \mathcal{F}$
- $A \in \mathcal{F} \Rightarrow \Omega \setminus A \in \mathcal{F}$
- $A_1, A_2, A_3 \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ (closed under countable unions)

Note: $\mathcal{F}$ is also closed under countable intersection
Proof

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$$

$$\text{Since } A \in \mathcal{F} \Rightarrow \overbrace{\Omega \setminus A}^{A^c} \in \mathcal{F}$$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i^c \in \mathcal{F}$$

$\Rightarrow$ Again this union is also a set in $\mathcal{F}$; and so is its complement

$$\bigcup_{i=1}^{\infty} A_i^c \in \mathcal{F} \Rightarrow \left(\bigcup_{i=1}^{\infty} A_i^c\right)^c \in \mathcal{F} \xrightarrow[\text{De Morgan's Laws}]{} \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$$

Examples of σ-algebras

~~Let $\Omega \neq \emptyset$~~

~~1)~~

(a)

1) $\{\emptyset, \Omega\}$ $\rightarrow$ Trivial σ-algebra

2) $\{P(\Omega)\}$ $\rightarrow$ discrete σ-algebra

3) Let $A \subseteq \Omega$ then $\{\emptyset, A, A^c, \Omega\}$ is a σ-algebra

$$\emptyset^c = \Omega \quad , \quad \Omega^c = \emptyset$$
$$(A)^c = A^c \, , \quad (A^c)^c = A$$
$$A \cup A^c = \Omega$$
$$\emptyset \cup A = A$$
$$\emptyset \cup A^c = A^c$$

and so on ...

σ-Algebras generated by a set

Let $T \subseteq P(\Omega)$

Then, there exist a unique smallest σ-algebra which contains every set in T.
$\hookrightarrow$ This is called the σ-algebra generated by T $(\sigma(T))$

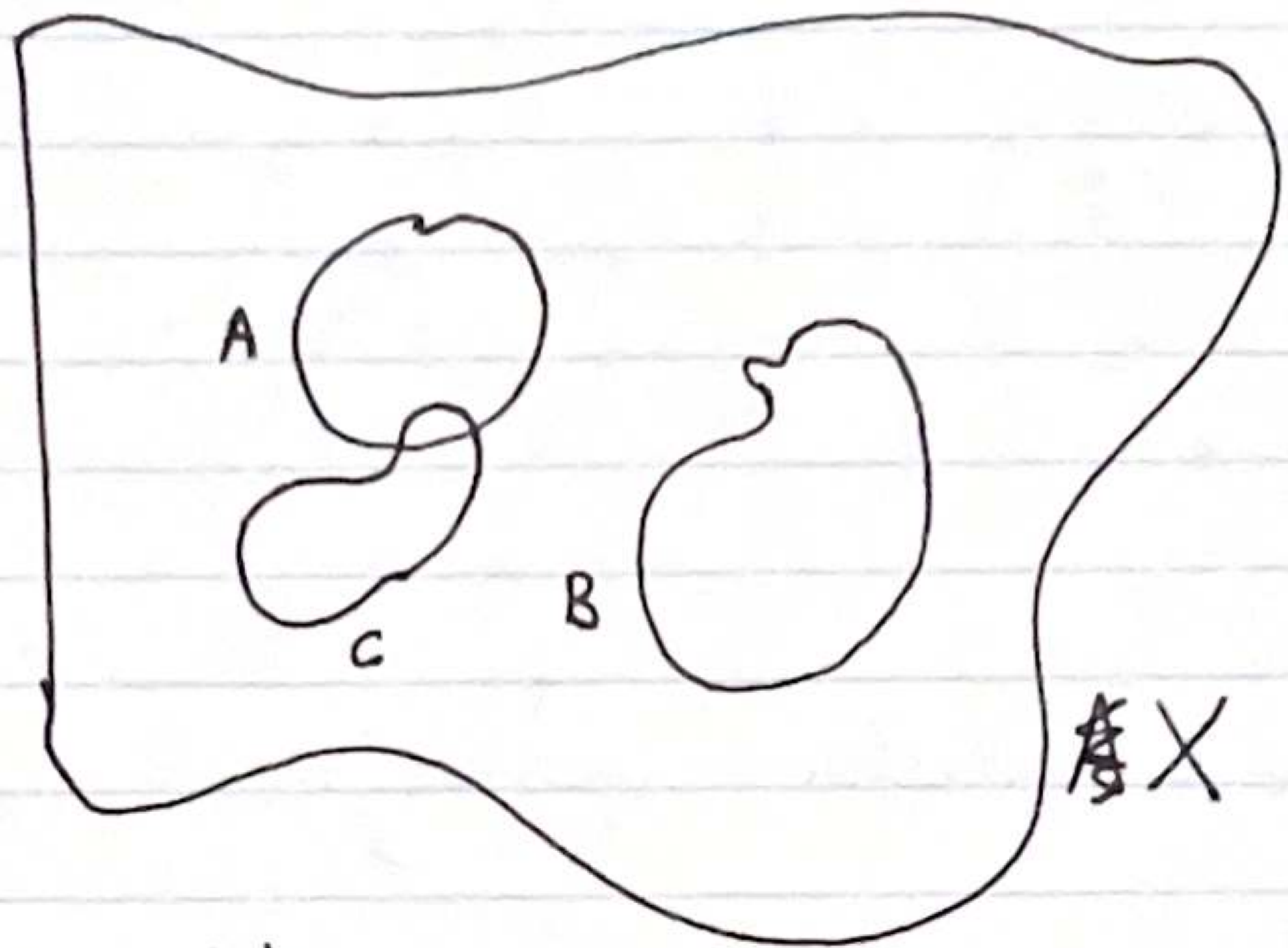
$\sigma(T) = \{$ Countable union & intersections of the complement of the elements of $T\}$

subsets

Topology.

I think I covered metric space defn and some elementary point-set defn before. I also covered them for Measure theory I think.

We have sets

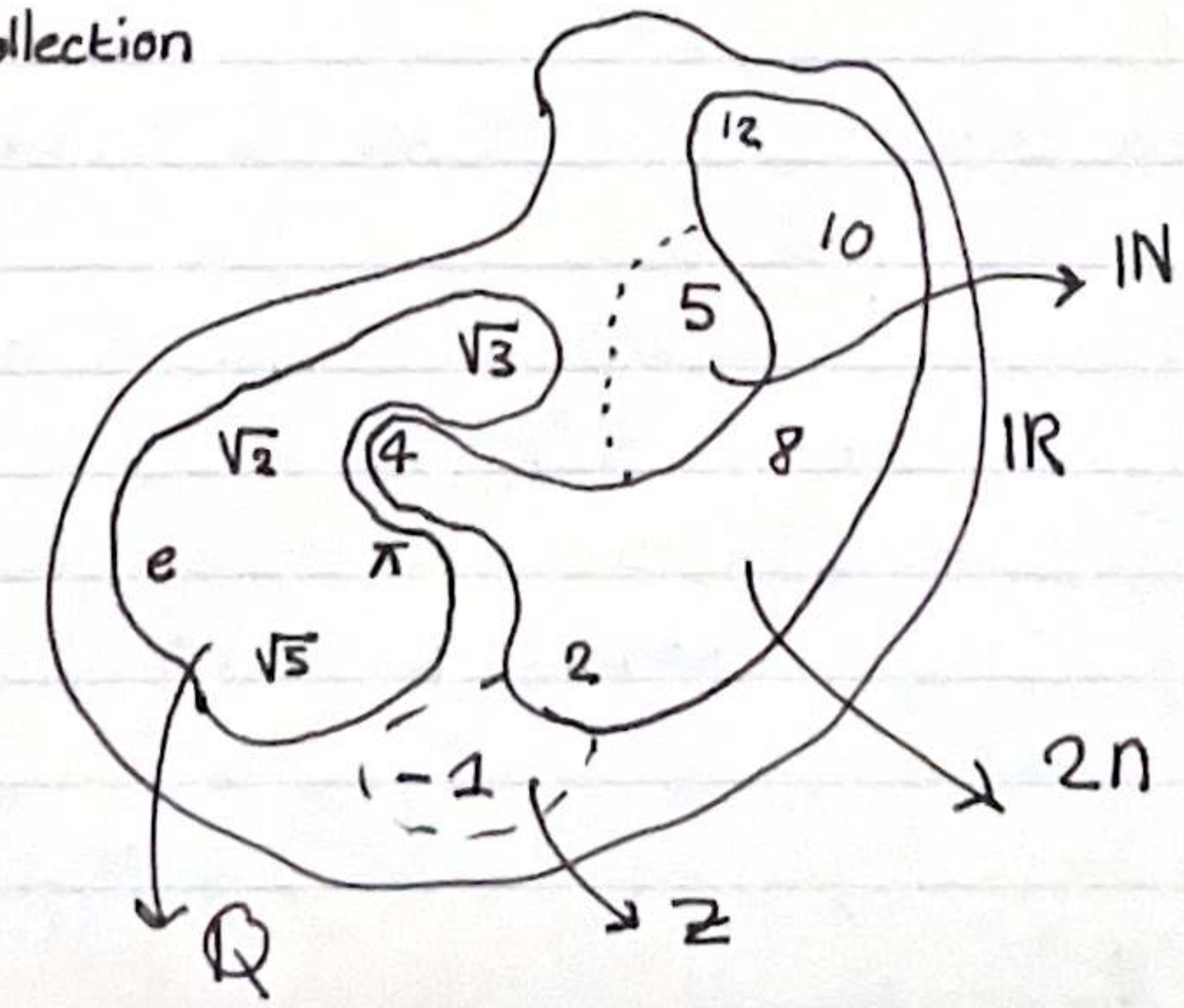


These are elements grouped by membership, forming a set. It can be anything from the set of animals to the set of real numbers.

New sets can be formed by union, intersection, set difference, complement.

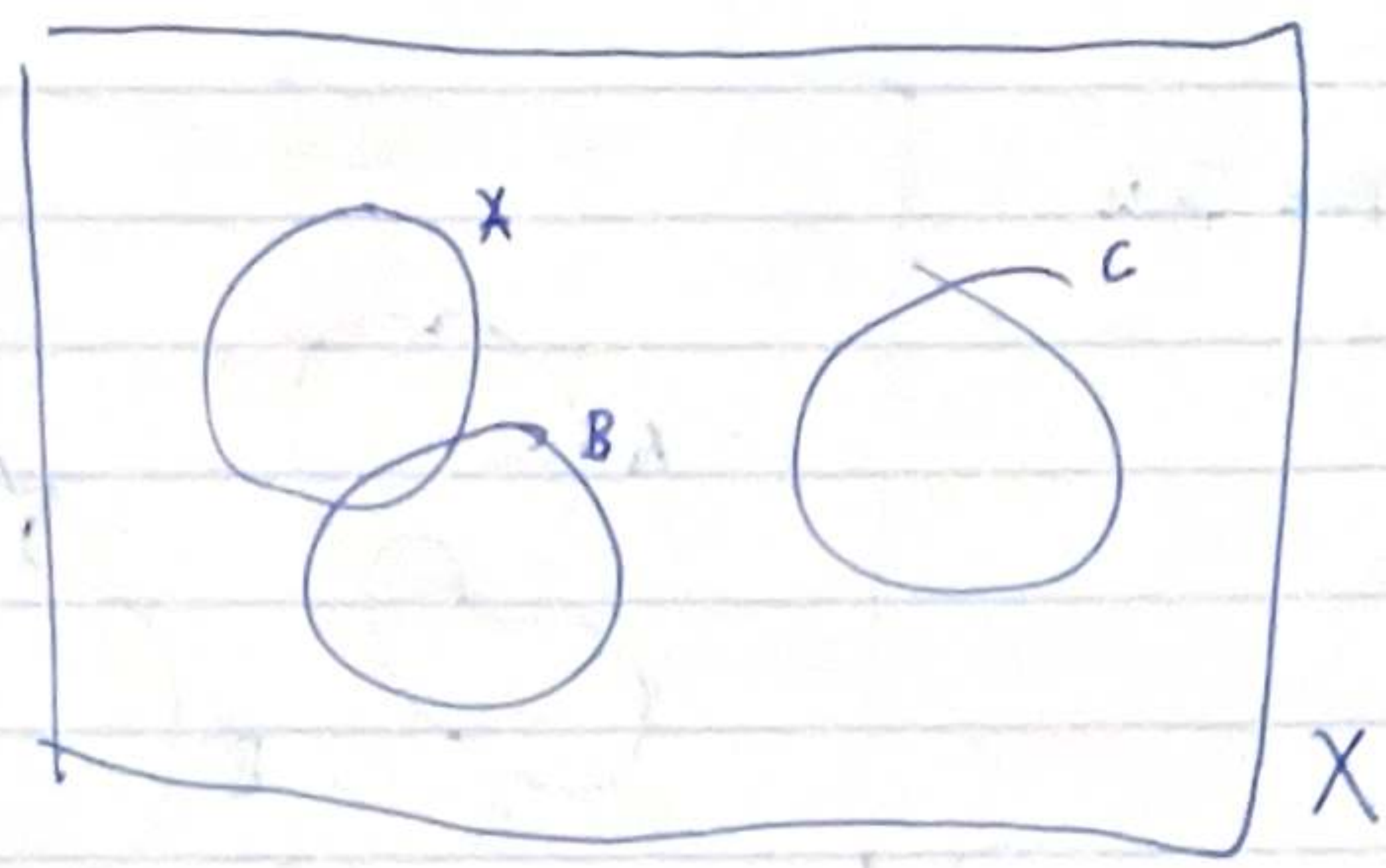
Order Theory: $a < b$, $a, b \in X$ (You can impose structure and add order to a set, by classifying its elements according to a relation).

These are just collections of objects.
You can have the set of $\mathbb{Q}$ in $\mathbb{R}$ or the set of even numbers.
It is just a collection



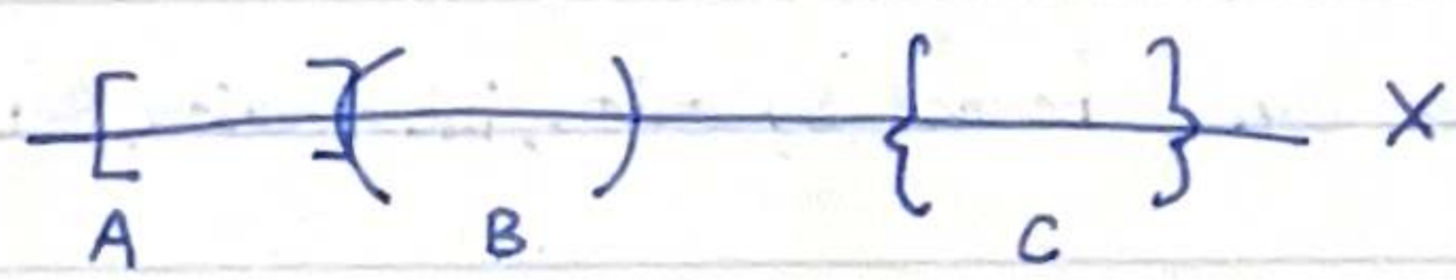
Meat - Space

Consider a set of subsets.



(X is a collection of subsets)

A has all elements with a common property, So do B & C.



There can be union & intersection and all other elementary set operations.
~~Sets like these define a "closed" collection of elements.~~

That's the most basic way of viewing objects. Categorization into elements sharing a common property.

What about another way of viewing the collection?

Motivation: I have intersection which can measure the overlapping of points.
The Union sum of that intersection could interpret proximity in a meaningful way?

This contributes to the concept of nearness for each subset.

$$\left|\bigcup^{x}\bigcap_{x} \text{Points} \ni x\right|$$

$\mathcal{T} \subseteq \mathcal{P}(X)$ is a topology on X if:

$\emptyset, X \in \mathcal{T}$

$\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$ for any subcollection (arbitrary union)

$\bigcap_{i=1}^{n} U_i \in \mathcal{T}$ for any finite subcollection.

Topology Axioms required

for the algebra of open sets:

$\emptyset$ & X form part of $\mathcal{T}$ and are defined as open.

Arbitrary unions of open sets are open

If we can cover a space using open parts, then the whole cover/space is open too

Allow cover for infinite space like $\mathbb{R}$ with union of small open sets to make the whole thing open.

Why arbitrary → union index can be finite, infinite, uncountable finite intersection requirement.

Consider $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$

$$\bigcap_{n=1}^{K} A_n = \bigcap_{n=1}^{K} \left(-\frac{1}{n}, \frac{1}{n}\right)$$

$A_1 = (-1, 1)$
$A_2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$
$A_3 = \left(-\frac{1}{3}, \frac{1}{3}\right)$

$\cdots A_3 \subset A_2 \subset A_1$

Each subsequent set is contained in the other.

$\vdots$

$A_3 \cap A_2 \cap A_1 = \left(-\frac{1}{3}, \frac{1}{3}\right)$ Smallest overlap

$\vdots$

It keeps going to the smallest finite case k

$$\Rightarrow \bigcap_{n=1}^{k} \left(-\frac{1}{n}, \frac{1}{n}\right) = \left(-\frac{1}{k}, \frac{1}{k}\right)$$

$\lim_{n \to \infty} a_n$

$\lim \{a_n\}_{k=0}^{\infty}$

An infinite union is the limit of a sequence of finite union

We can't take limit of a set; $\left(-\frac{1}{k}, \frac{1}{k}\right)$ is a set.