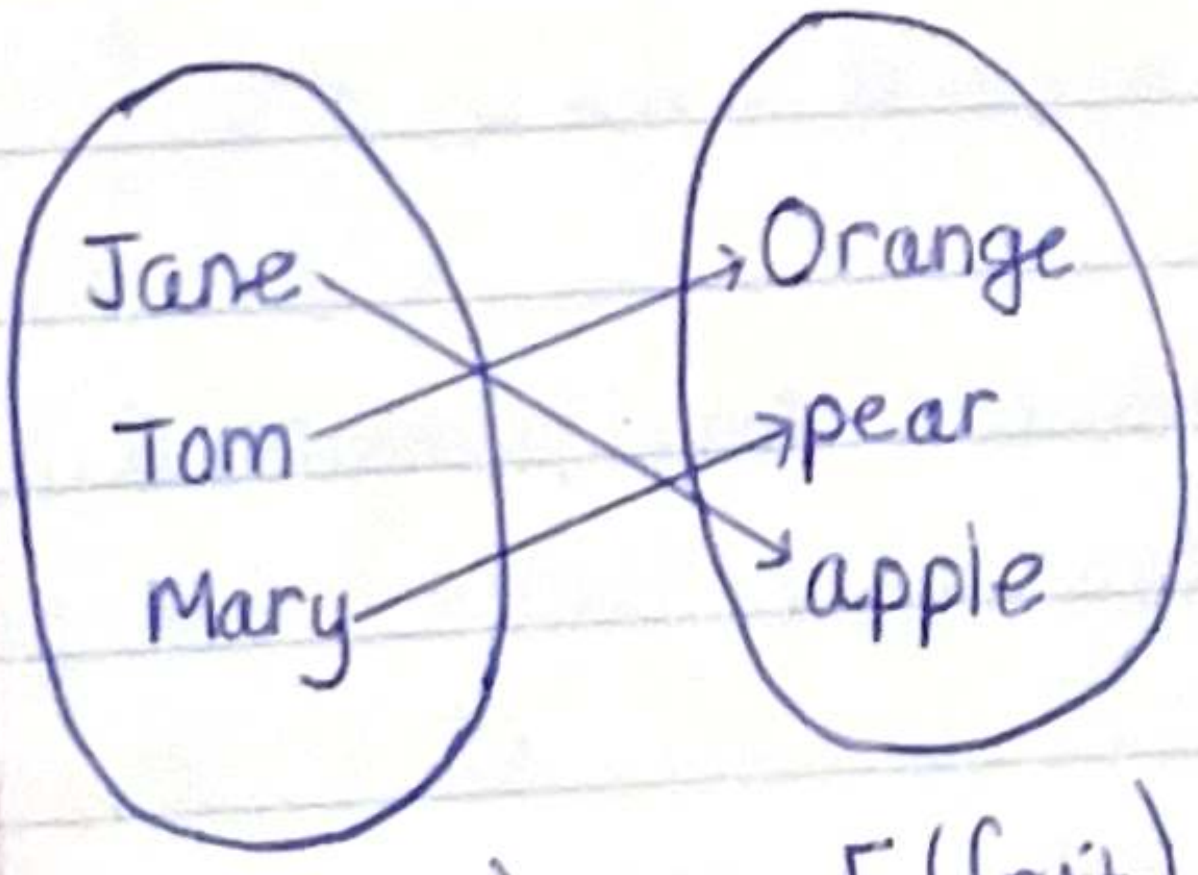
Binary Relations

Relations

- It is a way of relating an element of a set X to an element(s) of another set Y. (Intro to functions).

Suppose we are given the following
Jane has apple
Tom has Orange
Marco has pear

We can construct 2 sets as follows:



P (people) F (fruit)

Let "has" be a relation between the 2 sets.

The above relations can also be written as follows:

(Jane, apple)
(Tom, Orange)
(Mary, Pear)

These elements belong to the set of P x F (Cartesian Product.)

$P \times F = \{(J,a), (J,o), (J,P), (T,a), (T,o), (T,P), (M,a), (M,o), (M,P)\}$

(R)

It is ← a set

We can therefore define Binary Relation as follows:

- A binary Relation (R) over sets X and Y is a Subset of the Cartesian Product $X \times Y$

Where $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$

If $(x,y) \in R$ we say xRy.

- The set X is called the domain or set of departure of R and the set Y the codomain or set of destination of R.
- The statement (x,y) in R reads "x is R-related to y" and is denoted by xRy

~~Bin rel~~ set $= \{(x,y) \mid xRy\}$

- The domain of definition or active domain of R is the set of all x such that xRy for at least one y.
- The codomain of defn or active codomain, image, range of R is the set of all y such that xRy for at least one x.

We shall concern ourselves more with endorelations (homogeneous relations.)

Homogeneous Binary Relations.

- A binary relation over a set X is a Subset of the Cartesian Product $X \times X$.

eg. xGy : x is greater than y where $(x,y) \in \mathbb{Z}$

$(4,3)$ $4 > 3$ ✓

$(6,2)$ $6 > 2$ ✓

$(1,7)$ $1 \not> 7$ X

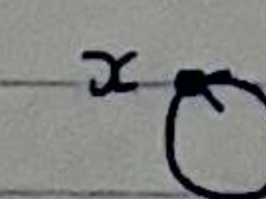
$(2.1,1)$ Vacuous (No ~~conclusions~~ claims can be made)

(Common) Properties of Binary Relations

$x,y \in \mathbb{N}$

1) Reflexive

$\forall x \ (\in X) \ xRx$



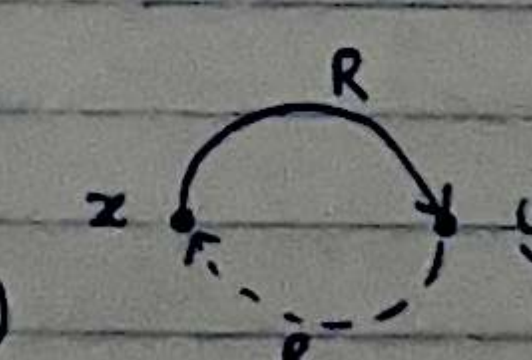
ie. x is related to itself

eg. equal relation (=)

Suppose $4 \in X$ then $(4,4) \in R$ as $4 = 4$

2) Symmetric

$\forall x \forall y \ (\in X) \ xRy \rightarrow yRx$



eg. Inequality Relation ($\neq$)

$4 \neq 3 \rightarrow 3 \neq 4$

Note: equality is also symmetric

ie $4 = 4 \rightarrow 4 = 4$

3) Transitive

$\forall_x \forall_y \forall_z \, (\in X) \quad xRy \,,\, yRz \rightarrow xRz$

eg. Strictly less than relation $(<)$

$3 < 5 \,,\, 5 < 7 \rightarrow 3 < 7$

$$\begin{tikzcd} x \arrow[r, bend left, "R"] \arrow[rr, bend right, "R"'] & y \arrow[r, bend left, "R"] & z \end{tikzcd}$$

Note : equality is also transitive

ie. $x = y \,,\, y = z \rightarrow x = z$

Note : The above are properties retrieved when studying a given relation.

eg. Let us study some relations and find their properties.

1) Less or equal to $(\leq)$

$1 \leq 1$

$\leq$ Reflexive ? Is $x \leq x$ true ? YES ✓

Symmetric ? Are $x \leq y$ and $y \leq x$ both true ?

Transitive ? $1 \leq 2 \,,\, 2 \leq 3 \rightarrow 1 \leq 3$ YES.

$a \leq b \,,\, b \leq c \rightarrow a \leq c$

If we define $x = y$ then $\leq$ is Symmetric.

(IN) Else $3 \leq 4 \rightarrow 4 \leq 3$?

then NO.

as Symmetry must hold $\forall x, y \in X$

Note : The properties of binary relations depend on how we define it.

Here is an example we encounter when constructing Integers. $\in \mathbb{N} \times \mathbb{N}$

Suppose we define $(m,n) \sim (p,q)$ iff $m + q = n + p$ where $(m,n),(p,q)$ ∧

ie $m - n = p - q$ * In $m,n,p,q \in \mathbb{N}$ ~

eg. $5 - 3 = 8 - 6 = 3 - 1$

ie. $\sim_{\circ} = \{(5,3),(8,6),(3,1),\dots\}$

Let us study the above relation.

$2 = (5,3) = (8,6),(3,1)$

Problem
Must we check for validity in the whole domain of departure or only in the domain of definition??

Is $\sim_{\circ}$ ~~transitive~~ Reflexive ? Same Entities

We know $(5,3) = (5,3)$

But $(5,3)$ also equals $(8,6),(3,1),\dots$?? Different Entities.

* But $(5,1)$

$(12,40)$ also Exist!!

Note : When we ask about the properties of $\sim$ we are talking about equality $(=)$ (as Stated).

This is called a quasi-reflexive relation ie. $\forall x,y \in X \mid x \sim y \Rightarrow (x \sim x \wedge y \sim y)$

In our example $(5,3) = (5,3)$

AND $(5,3) = (8,6),(3,1),\dots$

Also coreflexive?? ??

~~Is ~ Symmetric?~~

~~ie. Is $(m,n) \sim (p,q) \Rightarrow (p,q) \sim (m,n)$ true?~~

Review again

$(m,n) \sim (p,q)$ iff $\overbrace{m-n}^{x} = \overbrace{p-q}^{y}$ where $m,n,p,q \in \mathbb{N}$.

Reflexive

Is ~ ~~Symmetric~~?

Review This Problem!!

Defn

$\forall x \in X \;\; x \sim x$ is true.

In Equivalence Class

We know that for eg. $(3,0) \sim (3,0)$

But $\exists y \in X$ eg. $(5,2)$ ie. ~~y~~

$\Rightarrow$ quasireflexive.

2) $x - y \neq 0$

Is the relation reflexive, symmetric, transitive?

What does xRy translate to in this case?

$\rightarrow xRy$ iff $x - y \neq 0$

ie. $(x,y) \in R$ iff $x - y \neq 0$ $\Big\}$ $\forall x \forall y \in X \times X$ where ~~$X \in \mathbb{N}$~~ $X \rightarrow \{\mathbb{N}$

$\rightarrow X = \{x \mid x \in \mathbb{N}\}$

$x \in \mathbb{N}$

this describes a set.

Check

1) Reflexive?

$xRx \rightarrow x - x \neq 0$ ↯ as Any number minus itself equals zero.

$\Rightarrow$ NOT Reflexive.

2) Symmetric?

Different Nos $4 - 3 \neq 0$ and $3 - 4 \neq 0$ Both are not equal to zero

Same Nos $4 - 4 \neq 0 \rightarrow 4 - 4 \neq 0$? $\Rightarrow$ Symmetric.

Given $4 - 4 \neq 0$ p

$x - x \neq 0$ is symmetric q

$p \rightarrow q$

What is the truth value of $p \rightarrow q$

If $4 - 4 \neq 0$ then $x - y \neq 0$ is Symmetric.

hypothesis is false. ie. $4 - 4 \neq 0$ is false.

$\Rightarrow x - y \neq 0$ is Symmetric.

The Conditional statement is defined as follows:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth value

ie. $p \rightarrow q$ is only false when the hypothesis (p) is true and the conclusion (q) is false.

Note: The statements need not to be intuitively valid.

3) Transitive?

xRy , $yRz \rightarrow xRz$?

$5-2 \neq 0$ $3-1 \neq 0 \rightarrow 5-1 \neq 0$

In this case it is transitive

<u>But</u>

Consider

$3R4$, $4R3 \rightarrow 3R3$ ~~$\neq 0$~~

$3-3 \neq 0$ ↯ $\Rightarrow$ Not transitive.

~~$\Rightarrow$ Not transitive (False Con~~

We Search for a Possible Contradiction

~~The following Questions are Raised.~~

NOTE:

1) ~~For the properties of binary relations they must hold $\forall x \forall y \forall z \in$ Domain of definition (eg. $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$)~~ ~~$X\mathbb{N}$ $X\mathbb{R}$ $X\mathbb{Q}$ $X\mathbb{Z}$~~

~~ie. For Reflexivity~~

~~xRx must hold $\forall x \in \mathbb{N}$ for R to be a reflexive relation.~~

~~xRx must hold $\forall x \in$~~

Binary Relations

It is a subset of $X \times X$.

- A binary relation over a set X with itself is a Subset of $X \times X$.

- If an element (x, y) in $X \times X$ is also present in the binary relation subset we say xRy (x is R-related to y) or $x \sim y$.

- Binary relations are ways 2 elements ~~sets~~ of X relate to each other. Usually the set of ordered pairs $X \times X$ is considered.

- If an element of $X \times X$ (ie. 2 elements of X) satisfy a certain chosen relation then they are put in the subset R of $X \times X$.

- To choose elements of $X \times X$ to put in R we must check all elements of $X \times X$.

- We do so by checking all elements of $X \times X$ for the validity of a chosen relation R.

* The relation R holds for the elements in the subset R but its properties is universal for the whole set (including the subset)

- When checking, the properties of the relation R, it must ~~be valid~~ Universal and the same for all the elements in $X \times X$.

- Some examples of relations are $\leq, =, >, \neq, \ldots$ (NOT $+, -, \times, \circ, \ldots, |, \sqrt{\ }$)

These are relations. They compare 2 elements of X.

The arithmetic ~~operations~~ operators are NOT relations

$\boxed{\text{When we say } x \geq y \text{ all } (x, y) \text{ where } x \geq y \text{ is valid, belong to the subset } R}$

~~In this case the xRy is $\geq$??~~ $R \Leftrightarrow$

ie. The relation R is valid for all $x \geq y$ and we write xRy iff $x \geq y$ or $x \sim y$

$\boxed{\begin{array}{c} (x, y) \\ \downarrow \quad \downarrow \\ x R y \end{array}}$

Examples revisited

We must compare ordered pairs $\mathbb{N} \times \mathbb{N}$

1) Less than or equal to ($\leq$) on $\mathbb{N}$

ie. Find the properties of the relation R where xRy iff $x \leq y$

We remember that the properties of the relation is universal for all pairs of numbers.

1) Reflexivity : $\forall x \; xRx$ ie. All diagonals of the Cartesian product have valid R. (In this case R is $\leq$ relation) — x is related to x

eg. $5R5 \rightarrow 5 \leq 5$ (True) — ie. x is related to itself.

Non-example : les strict less than sign ($<$)

2) Symmetry : $\forall x \forall y \; xRy \rightarrow yRx$

We need to find the truth value of the Conditional statement :

If xRy then yRx. Truth implies Symmetry

xRy	yRx	
$5 \leq 5$	$5 \leq 5$	
True	True	$\Rightarrow$ True.

xRy	yRx	
$2 \leq 5$	$5 \leq 2$	$\Rightarrow$ False conclusion

$\Rightarrow$ No symmetry.

We have found a contradiction to the proposition that xRy is symmetric. Hence we conclude that xRy is not symmetric.

3) Transitivity : $\forall x \forall y \forall z \quad xRy, yRz \rightarrow xRz$

We shall not bother with trivial cases but we shall try to find a contradiction to symmetry.

$2R5, \; 5R7 \rightarrow 2R7$

$2 \leq 5 \quad 5 \leq 7 \rightarrow 2 \leq 7$ (True)

$\Rightarrow$ Transitive (No contradiction was found).

2) $x - y \neq 0$ on $\mathbb{Z}$ — We draw no conclusion on $\mathbb{N}$ as $\mathbb{Z}$ are required for some cases. — Biconditional def$^{\underline{n}}$.

ie. Find the properties of the relation R where xRy iff $x - y \neq 0$

$(\mathbb{N}, \mathbb{N})$

The ordered pairs (x, y) which satisfy the relation $x - y \neq 0$ are put in the subset R (of $\mathbb{N} \times \mathbb{N}$) [but each arguments are in $\mathbb{N}$ ie. (x, y) — $\mathbb{N}$, $\mathbb{N}$]

But we remember that the properties of R is universal for all elements. (It could have a certain property for some specific element BUT we are looking out for universal properties)

IMPORTANT: The relation R could be valid for a specific subset of elements BUT, its properties must be universal across all elements.

1) Reflexivity $\forall x \; xRx$ is true.

$x - x \neq 0$ (False)

$\Rightarrow$ NOT Reflexive.

Additional iff

$\boxed{\text{If } xRy \text{ then } x - y \neq 0}$

If $x - y \neq 0$ then xRy

Find out how to use Conditional statements for this.

2) Symmetry. $xRy \rightarrow yRx \ \forall x \forall y$

$x - y \neq 0 \rightarrow y - x \neq 0$

We need to find the truth value of the statement

If $x - y \neq 0$ then $y - x \neq 0$ (Not biconditional)

(5,5)

$5-5 \neq 0 \rightarrow 5-5 \neq 0$

If we look at the T-table of a cond. Statement, if hypothesis is false then truth value = true.

False False

(True)

(4,3)

$4-3 \neq 0 \rightarrow 3-4 \neq 0$

True True

(True)

No contradiction found $\Rightarrow$ R is symmetric.

No false conclusion found.

3) Transitivity : $\forall x \forall y \forall z \ xRy, yRz \rightarrow xRz$

(5,3) (3,2) $5-3 \neq 0$, $3-2 \neq 0 \rightarrow 5-2 \neq 0$

(True)

One of the hypothesis is false. use a substitution rule to break, this?? → We aim to find a contradiction via a false conclusion.

(5,5),(5,2) $5-5 \neq 0$, $5-2 \neq 0 \rightarrow 5-2 \neq 0$

False True → $5-2 \neq 0$

True.

(True)

$3-2 \neq 0$, $2-3 \neq 0 \rightarrow 3-3 \neq 0$

True True False Conclusion

$\Rightarrow$ No Transitivity.

Intuition behind Conditional & Biconditional Statements

I wear a coat if and only if it rains.

I wear a coat

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
It rains → T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

If it does not rain I wear a coat is true. [I can wear a coat w/o rain!!]

If it rains I do not wear a coat is false

If I wear a coat then it does not rain is false. Wearing coat has no effect on rain.

If it does not rain I do not need to wear a coat.

If I do not wear a coat then it rains is true. Wearing coat is independent of raining.

3) $(m,n) \sim (p,q)$ iff $m-n = p-q$ where $m,n,p,q \in \mathbb{N}$

✓ 1) Reflexivity

$(m,n) \sim (m,n)$ (True)

$m-n = m-n$

We do not concern ourselves with this only that reflexivity is satisfied!!

~~(5,2)~~

$5-2 = 3-0$ (True)

But $5-\cancel{2} = 3-0 = 2$

$\Rightarrow 2 = 2 \Rightarrow$ Reflexive.

And $(3,0) \sim (3,0)$

$\Rightarrow$ quasi-reflexive (Needs to be reflexive in first place)

✓ 2) Symmetry

$(m,n) \sim (p,q) \rightarrow (p,q) \sim (m,n)$

$m-n = p-q \rightarrow p-q = m-n$

$\Rightarrow$ Symmetric

3) Transitivity.

✓ $(m,n) \sim (p,q)$ $(p,q) \sim (r,s) \longrightarrow (m,n) \sim (r,s)$

$m-n = p-q$ $p-q = r-s \Rightarrow m-n = r-s$

$\Rightarrow$ Transitive.

This is same for $x \sim y$ iff $x = y$

✱ The equal sign creates an Equivalance Class.

↘ Any relationship satisfying Symmetry, reflexivity and transitivity is an equivalance class.

4) $x R y$ iff $x \neq y$

Reflexive?

$x \neq x$ (false)

Symmetric?

$x \neq y \rightarrow y \neq x$ (True)

Transitive

$x \neq y$ $y \neq z \rightarrow x \neq z$?

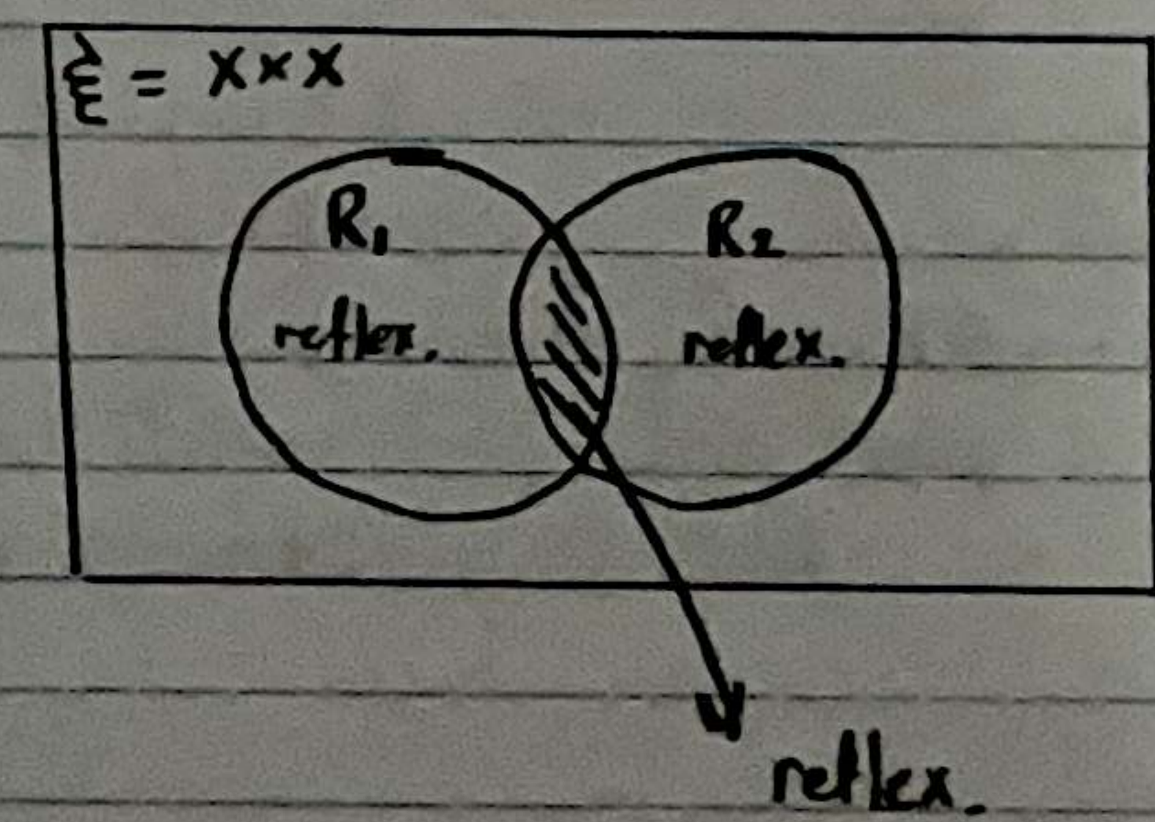
Find counter example to lead to false conclusion.

$x=2, y=1, z=2$

$2 \neq 1$ $1 \neq 2 \rightarrow 2 \neq 2$ (False)

<u>Some examples on Relations</u>

Prove: If R_1 and R_2 are reflexive than $R_1 \cap R_2$ is reflexive.



Suppose we pick an arbitrary (x,x) in both R_1 & R_2

$\Rightarrow (x,x) \in R_1 \cap R_2$

$\Rightarrow R_1 \cap R_2$ is ~~symmet~~ reflexive.

x is never related to itself.

If $\forall x, (x,x) \notin R$ then R is irreflexive.

(i) Give an example where R is irreflexive, transitive, but not symmetric.

↳ Nothing is related to itself $\rightarrow x < y$ or $x > y$

↳ No symmetry $\rightarrow x < y$ or $x > y$

↳ "Order" is induced by transitivity $\rightarrow x < y, y < z \Rightarrow x < z$

$\rightarrow <$ (less than sign)

(ii) Show that if R is ~~&~~ transitive & Symmetric, it cannot be irreflexive.

$xRy \rightarrow yRx$

$xRy, yRz \rightarrow xRz$

Assume xRy

By Symmetry yRx

$xRy, yRx \rightarrow xRx$ By transitivity

Reflexivity must be universal.

Equivalance Class

- When the elements of some set S have a notion of equivalance (formalised as an equivalance relation) defined on them, then one may naturally split the set S into equivalance Classes.
- These equivalance Classes are constructed so that the elements a and b belong to the same equivalance class if and only if they are equivalent.

This is universal for all elements of S.

Formally, given a set S and an equivalance relation on S, the equivalance class of an element a in the set $\{x \in S \mid x \sim a\}$ of elements which are equivalent to a.

- Equivalance relation is defined over all S (all elements of S) ie. the relation is reflexive, Symmetric & transitive $\forall x \in S$. Therefore elements may or may not be equivalent to another element.

eg. Some elements may be equivalent to $a, b, c, \dots \in S$. Grouping the equivalent elements together eg all elements equivalent to a form $\{x \in S \mid x \sim a\}$, to b form $\{x \in S \mid x \sim b\}$ and so on. In this way, the equivalance classes form a partition of S

grouping of elements in a non-empty subset.

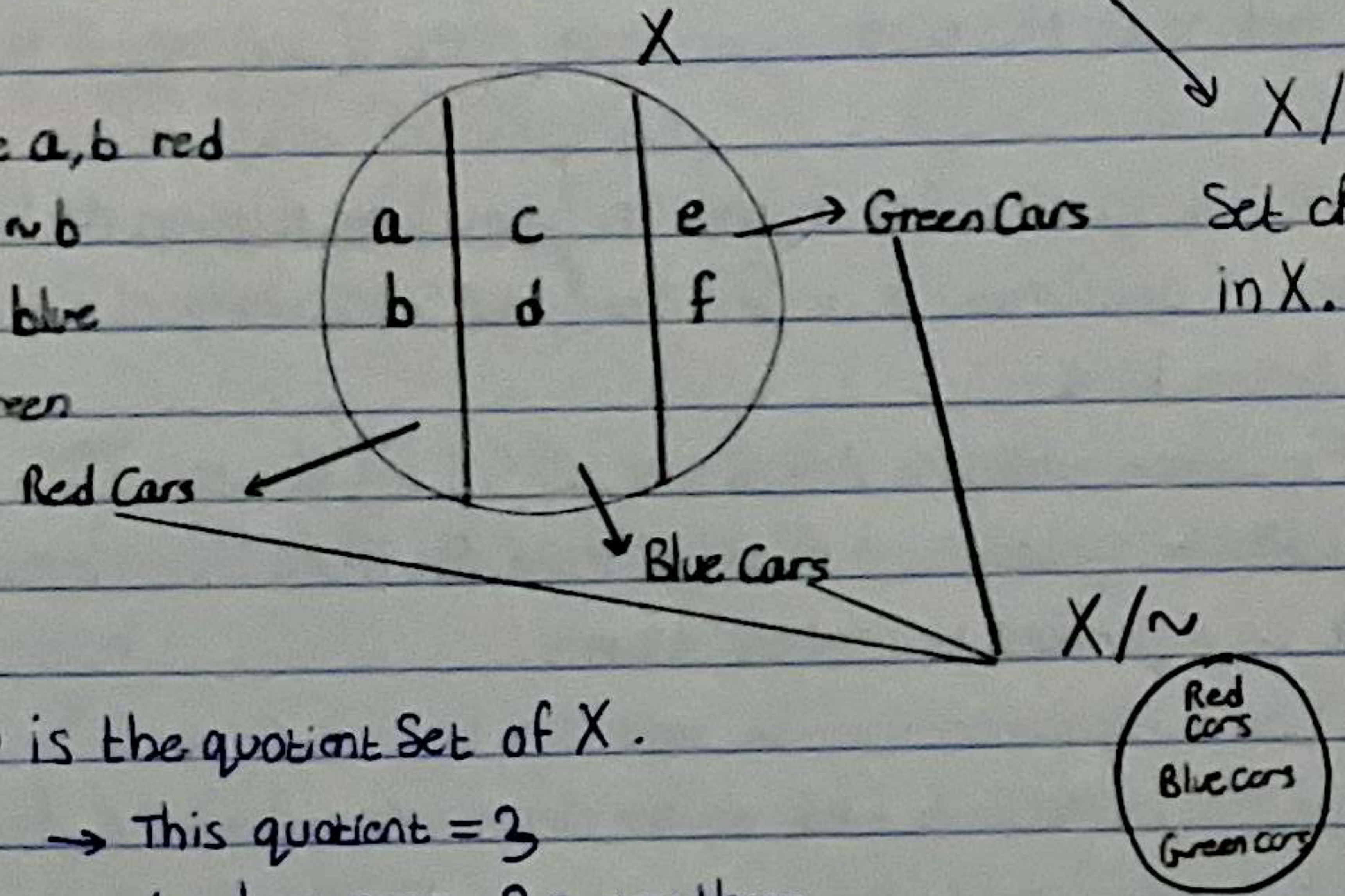
This partition - The set of equivalance classes - is sometimes called the quotient set or quotient space of S by $\sim$ and is denoted by $S/\sim$.

Examples

1) If X is the set of all cars, and $\sim$ is the equivalance relation "has the same colour" then one particular equivalance class consists of all green cars. $X/\sim$ could be naturally identified as the set of all car colours.

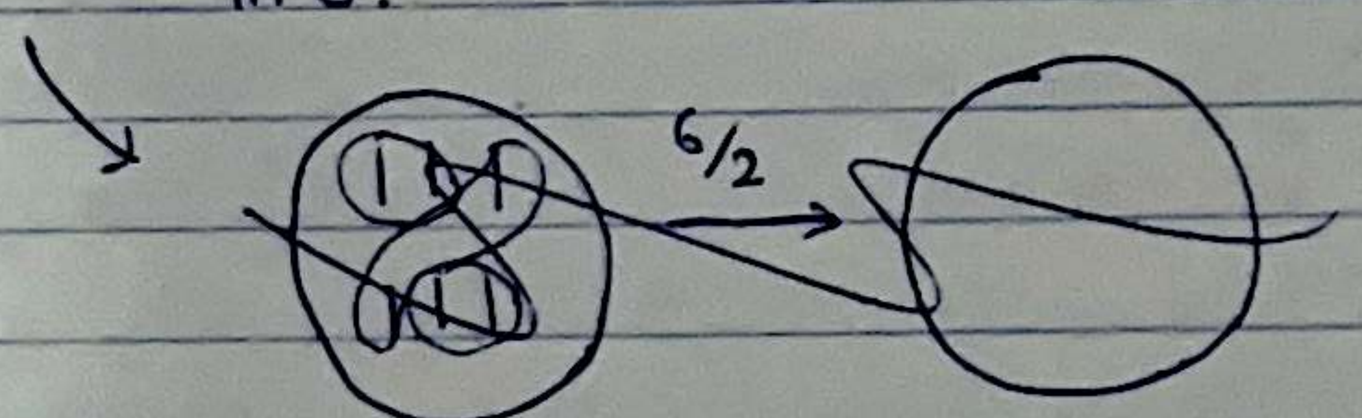
$X/\sim$ is the set of all eq. classes in X.

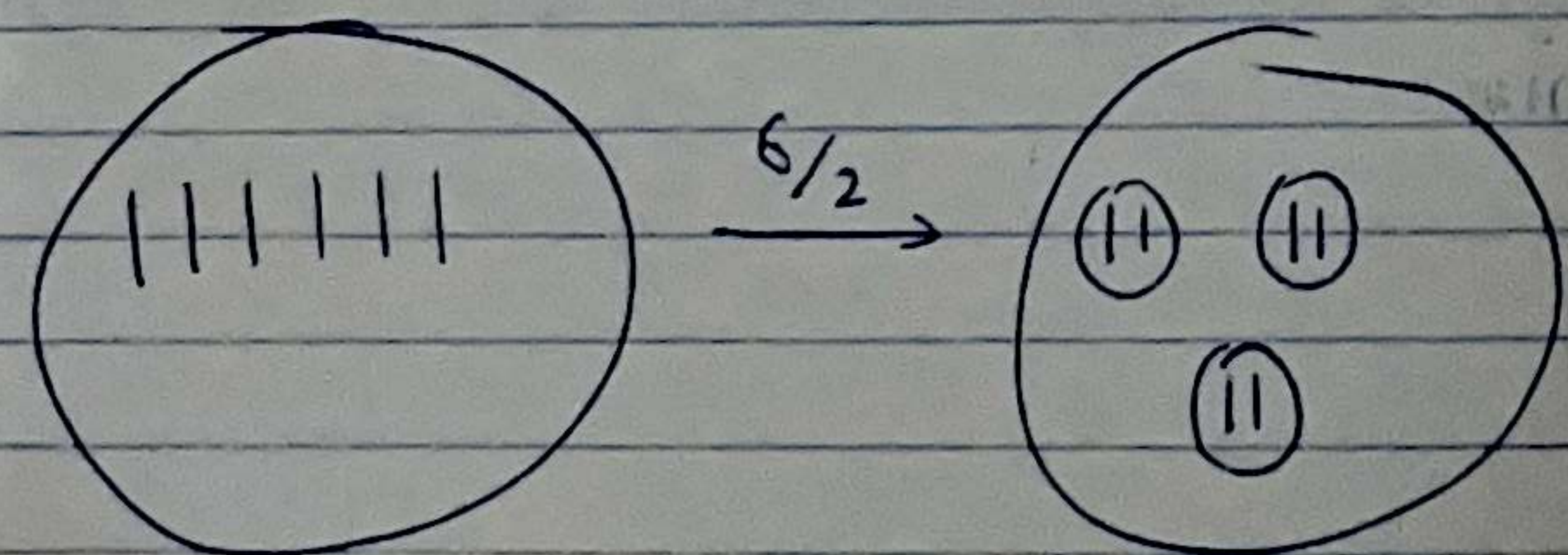
Suppose a, b red
$\Rightarrow a \sim b$
c, d blue
$e f$ green



$X/\sim$ is the quotient set of X.

$\cancel{6}/2 \rightarrow$ This quotient $= 3$
ie. how many 2s are there in 6.





2) If R is the set of all rectangles ~~with area A~~ in the plane and the equivalance relation $\sim$ "has the same area as". For each positive Real number A, there will be an equivalance class of all the rectangles that have area A.

3) Consider the modulo 2 equivalance relation on the set of integers $\mathbb{Z}$, such that $x \sim y$ if and only if their difference $x - y$ is an even number. This relation gives rise to exactly 2 equivalance classes: One class consists of all even numbers, and the other class of all odd numbers. Using square brackets around one member of the class to denote an equivalance class under this relation*, $\underbrace{[7], [9], [1]}_{\text{All of these form the equivalance class of odd numbers in } \mathbb{Z}}$ all represent the same element of $\mathbb{Z}/\sim$.

Note: $[a] \underset{\text{or}}{=} \{x \in X \mid a \sim x\}$
$[a]_\sim$

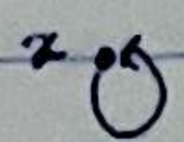
* ie. $[7]$ denotes all the elements in the same equivalance class as 7 ie $[7]$ denotes the equivalance class of 7 and $[7]$ is same as $[9]$ as both are in the class of odd numbers.

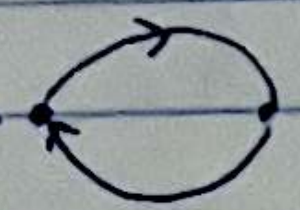
All of these form the equivalance class of odd numbers in $\mathbb{Z}$ and are the same in $\mathbb{Z}/\sim$ as the latter "sees" them as solely odd numbers.

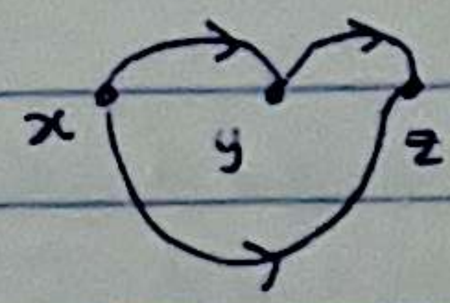
Note: The equivalance class is w.r.t the stated equivalance relation.

Equivalance Relations → The relation is most commonly "=" sign.

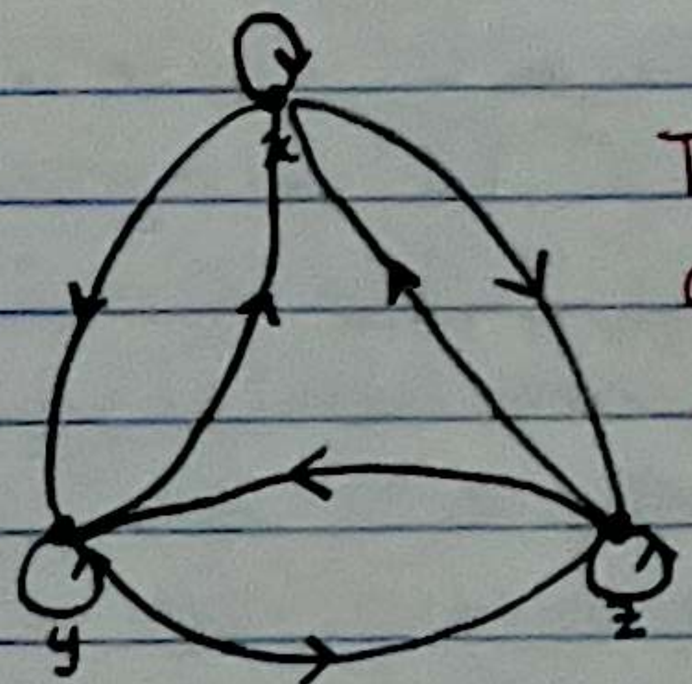
An equivalance relation is one which is:

1) Reflexive : $\forall x \; xRx$  Arrow denotes Relation.

2) Symmetric : $\forall x \forall y \; xRy \rightarrow yRx$ 

3) Transitive : $\forall x \forall y \forall z \; xRy, yRz \rightarrow xRz$ 

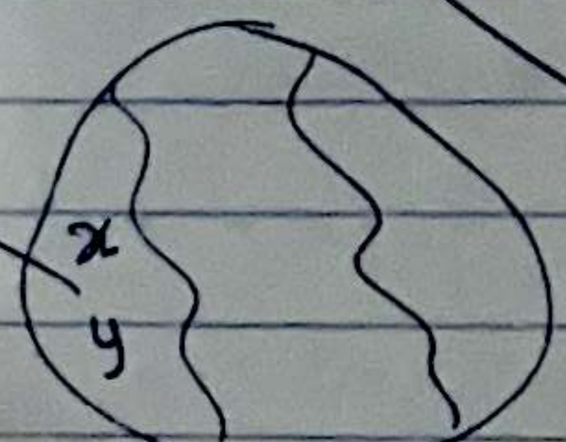
Suppose we have 3 equivalent elements $(x = y = z)$



This diagram states that all 3 elements are equivalent to each other.

Suppose R is an equivalance relation on some set X
For each $x \in X$, let $[x] = \{y \mid xRy\}$

Then $[x] = [y]$
↳ Both x & y are in the same equivalance class.



Same eq-class
ie $[x] = [y]$

Specific case
$[a] = \{b\}$ ← The only element in the equivalance class of a.
for a • → • b
aRb

OR $[x] \cap [y] = \emptyset$

Suppose aRb and cRb then $[a] = [c]$

Suppose $aRb \rightarrow \neg aRd$
$cRd \rightarrow \neg cRb$ } $a \not R d$ and $c \not R b$ where $\not R$ is not related.

Then a & ~~c~~ do not share the same equivalance relation hence Empty set.

→ New Relation : Anti symmetry.

defn $(aRb \wedge bRa) \rightarrow a = b$ The defn of anti symmetry says nothing about (a,a).

ie. If 2 objects are both related to each other then they are the same objects.

Example : $(\leq)$ (operator??)
$x \leq y$ and $y \leq x$ are both ~~true~~ valid then $x = y$
↳ only way this makes sense

: $(\subseteq)$
$A \subseteq B$ and $B \subseteq A$ then $A = B$
All of B is contained with A and vice versa.

Partial Order $x \preceq y$

↳ Not an equivalance relation but another type of relation.

- Antisymmetric.
- Reflexive
- Transitive.

Some types of partial Orders

- $x \leq y$

↳ Suppose we have

$2 \leq 2$ reflexive

$2 \leq 4$ and $4 \leq 5 \to 2 \leq 5$ transitive.

$x \leq y$ and $y \leq x \to x = y$ antisymmetric

- Alphabetical order: $(a, b, c, \dots, z)$

$a \leq a$ (posn of letters)

$a \leq b, b \leq c \to a \leq c$

$x \leq y, y \leq x \to x = y$

$a \preceq b \preceq c, \dots$ Poset.

Equivalance Relations (Continued.)

Defn : Equivalance class of $x \in X$

$$E_x = \{ y \in X \mid yRx \}$$

We fix x and find all the ys equal to x. We find out x (ie. the value of x) is the only member for this relation.

Remember an equivalance relation is symmetric ie $xRy \Rightarrow yRx$

eg. 1 $E_x = \{ y \in X \mid y = x \} = \{ x \}$ (y is an arbitrary element)

$\{x\} = \{x, x, x, x, \dots\}$

eg. 2 $E_x = \{ y \in X \}_{yRx} = X$ ⟶ It denotes that the equivalance class of x is the entire set itself.

Remember xRy

NOTE: E_x is same as $[x]$

eg. 3

$$E_x = [x] = \{ y \in \mathbb{Z} \mid y - x = kn \}$$
$$= \{ \dots, x-n, x, x+n, x+2n, \dots \}$$

⟶ Set consisting of integers congruent to x mod n (aka $\bar{x}_n$)

*This set is known as the congruence class or residue class of integer x modulo n.

When n = 2 $E_0 = \{ \text{Evens} \}$ $E_1 = \{ \text{Odds} \}$

$= \{0, 0+2, 0+2(2), \dots\}$ $= \{1, 1+2, 1+2(2)\}$

- Like any congruence relation, congruence modulo n is also an equivalance relation.

Suppose we want the congruence class of an integer mod n.

① We have Equivalance class of an integer $x \in \mathbb{Z}$

We know $[x] = \{ y \in \mathbb{Z} \mid yRx \}$

② $E_x = [x] = \{ y \in \mathbb{Z} \mid y - x = kn \} \left(= \{ y \in \mathbb{Z} \mid y = x + kn \} \right)$ (Defn of modulus)

③ Suppose we want the congruence class of integer 5 mod 2 then $x = 5$ and $n = 2$

Then ~~$[5] = \{5 - 2(2), 5 - (2)$~~

$[5] = \{5 - 2(2), 5 - 1(2), 5 - 0(2), 5 + 1(2), 5 + 2(2), \dots\}$

$= \{\dots, 1, 3, 5, 7, 9, \dots\}$

③ ~~Suppose we want the congruence class~~

eg.3 Residue Class of Integer Modulo n.

- Like any Congruence ~~class~~, relation, Congruence modulo n is also an equivalance Relation.

Congruence

Given an integer $n > 1$ called modulus, two integers are said to be congruent modulo n if n is a divisor of their difference

ie. There is an integer k such that $a - b = kn$

Congruence modulo n is denoted : $a \equiv b \pmod n$

Example

$38 \equiv 14 \pmod{12}$

Because $38 - 14 = 24$ which is a multiple of 12 or equivalently both 38 and 14 have the same remainder 2 when divided by 12.

- The defn of Congruence can be extended to negative integers

$-8 \equiv 7 \pmod 5$ ie. $-8 - 7 = -15$ which is a multiple of 5.

$2 \equiv -3 \pmod 5$ ie. $2 - -3 = 5$ which is a multiple of 5.

Congruence Classes

- Like any congruence relation (eg. Similar triangles, congruence class in Set of all planar triangles), *Congruence modulo n is also an equivalence relation (eg. equivalence Class of all green cars in the set of cars of different colour.)

The equivalance class of the integer a modulo n is denoted by $\overline{a}_n$ is the set $\{\dots, a-2n, a-n, a, a+n, a+2n, \dots\}$

This set consisting of all the integers congruent to a modulo n is called the congruence class / residue class or residue of the integer a, modulo n.

When the modulus is known, the class may also be denoted $[a]$

Example

Find the residue class of the integer 5 modulo 3.

↳ We must find the set of integers which have the same remainder as 5 when divided (by multiples of) 3. We must find for y s.t $y \equiv 5 \pmod 3$

In general we have an integer $x \in \mathbb{Z}$

We know that the equivalance class of x is $[x] = \{ y \in \mathbb{Z} \mid yRx \}$

↳ Set of all integers which is equivalent to x.

(residue class)

We also know that the equivalance class of the integer x modulo n is

$\overline{x}_n = \{ y \in \mathbb{Z} \mid \cancel{x-y}\ y-x = kn \} = \{ y \in \mathbb{Z} \mid y = x + kn \}$

$= \{ \dots, x-2n, x-n, x, x+n, x+2n, \dots \}$

We have

$[5]_3 = \{ \dots, 5-2(3), 5-(3), 5, 5+3, 5+2(3), \dots \}$ (+3, +3, +3)

$= \{ \dots, -1, 2, 5, 8, 11, \dots \}$

↳ This set consists of all the integers congruent to 5 modulo 3.

This is why they form an equivalance/congruence class.

~~$\frac{5}{3} = 2 \text{ (rem)}$~~ ✗

~~$\frac{-1-5}{3} = -2 \text{ (multiple)}$~~

↳ ~~$-1 - 5 = -6$ (multiple of 3)~~

~~$2 - 5 = -3$ (multiple of 3)~~

- Two integers are said to be congruent modulo n if n is a divisor of their difference.

We must check for y in the set s.t $y \equiv 5 \pmod 3$

ie. $\frac{y-5}{3}$ must be whole | $\frac{-1-5}{3} = \text{whole}$ | $\frac{5-5}{3} = \text{whole}$ ✓ | $\frac{11-5}{3} = \text{whole}$ ✓

$\frac{2-5}{3} = \text{whole}$ ✓ | $\frac{8-5}{3} = \text{whole}$ ✓

Note: Suppose we have $a \equiv b \pmod n$ then $1 < n \leq b$.
If $n > b$ eg $a \equiv 5 \pmod 7$ this would not make sense as $5/7$ does not have a whole number as Remainder.

X ~~WRONG~~

From def$\underline{n}$ we need a s.t $\frac{a-5}{7}$ is an Integer (Whole Number)

Extension to negative numbers

Note: ~~5 mod 7 is different from 5 (mod 7)~~

Look for

Note: 5 mod 2 is different from 5 (mod 2)

Parentheses mean that (mod n) apply to both LHS & RHS.

We look for the unique integer $0 \leq a < n$ s.t $a \equiv b \pmod n$

In this case $a = 1$

In case 5 mod 7

We look for $\boxed{0 \leq a \leq 7}$ s.t $a \equiv 5 \pmod 7$

$a \to 5$

$\frac{5-5}{7} = 0$ Whole $\underline{No}$

a mod n generates an unique $\underline{no}$

b (mod n) generates a set of no_s

When we set $n = 2$ we can generate sets consisting of only even or only odd integers.

eg. $[0]_2 = \{\dots, -4, -2, 0, 2, 4, \dots\}$

$[1]_2 = \{\dots, 1-2(2), 1-2, 1, 1+2, 1+2(2), \dots\}$
$= \{\dots, -3, -1, 1, 3, 5, \dots\}$

$[5]_2 = \{\dots, 5-2(2), 5-2, 5, 5+2, 5+4, \dots\}$
$= \{\dots, 1, 3, 5, 7, 9, \dots\}$

$[6] = \{\dots, 6-2(2), 6-2, 6, 6+2, 6+4, \dots\}$
$= \{\dots, 2, 4, 6, 8, 10, \dots\}$

$[Even]_2 = \{Evens\}$

$[odd]_2 = \{Odds\}$

When $n=0$

$\Rightarrow \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$

For residue class of Integers modulo n, there are 2 things we could study.

1. The set of all congruence class of $\underline{integers}$ for $\underline{a}$ modulus n
   - Called the ring of integers modulo n and is denoted by $\mathbb{Z}/n\mathbb{Z}$

$\mathbb{Z}/n\mathbb{Z} \stackrel{def}{=} \{\bar{a}_n \mid a \in \mathbb{Z}\}$ for $n > 0$

$= \{\bar{0}_n, \bar{1}_n, \bar{2}_n, \dots, \overline{n-1}_n\}$

ie. $\mathbb{Z}/n\mathbb{Z} = \{[0]_n, [1]_n, [2]_n, \dots [n-1]_n\}$

Quotient Set

~~Non as $[n]_n = 0$, $n \bmod n = 0$~~ ??

One Element of this quotient set $\mathbb{Z}/n\mathbb{Z}$ represents a set of elements congruent mod n to the element representing the set.

Remember:

$[x]_n = \{\dots, x-2n, x-n, x, x+n, x+2n, \}$

When we set $n = 2$

$[0]_2 = [2]_2 = [4]_2 = \dots$

$[1]_2 = [3]_2 = [5]_2 = \dots$

$\Rightarrow \mathbb{Z}/n\mathbb{Z}$ becomes $\{[Evens]_2, [Odds]_2\}$

where $[Evens]_2 \neq [Odds]_2$

The set of all congruence class of $\underline{an}$ integer for $\underline{different}$ modulus n.

When we set $n=3$

$[0]_3 = \{\dots,-6,-3,0,3,6,\dots\}$

$[1]_3 = \{\dots,-5,-2,1,4,7,\dots\}$

$[2]_3 = \{\dots,-4,-1,2,5,8,\dots\}$

$[3]_3 = \{\dots,-3,0,3,6,9,\dots\} = [0]_3$

$[4]_3 = \{\dots,-2,1,4,7,10,\dots\} = [1]_3$

$[5]_3 = \{\dots,-1,2,5,8,11,\dots\} = [2]_3$

$[6]_3 = \{\dots,0,3,6,9,12,\dots\} = [3]_3 = [0]_3$

Conclusion

- When $n=2$ $\mathbb{Z}/_{2\mathbb{Z}}$ Consists of 2 Classes which are disjoint but their union make up $\mathbb{Z}$ again.
- When $n=3$ $\mathbb{Z}/_{3\mathbb{Z}}$ Consists of 3 disjoint classes whose union make up $\mathbb{Z}$ again.

⋮

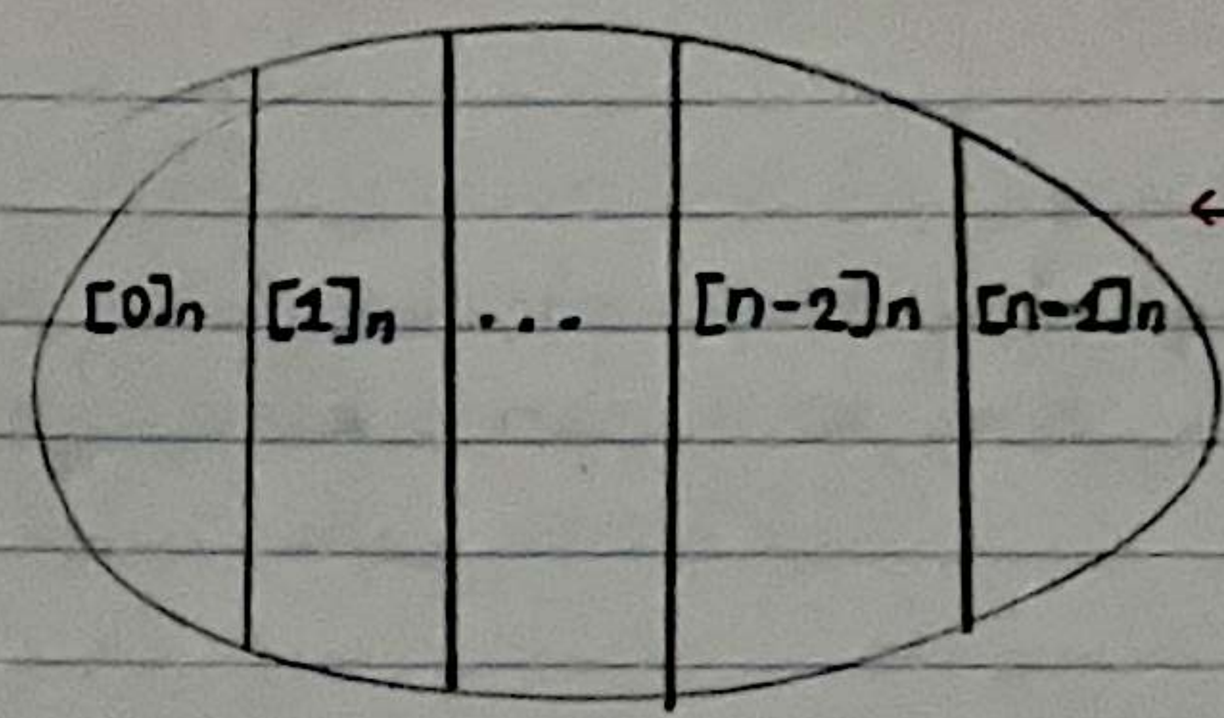
- When $n=n$ $\mathbb{Z}/_{n\mathbb{Z}}$ consists of n disjoint classes whose union make up $\mathbb{Z}$ again.

We can see that equivalance relation / Congruence modulo n relation on $\mathbb{Z}$ Sorts the elements of $\mathbb{Z}$ into n disjoints sets based of the relation ie. all elements congruent to $0 \pmod n$ go in $[0]_n$. We can also see that for some n, some classes share the same elements as others ie. $[0]_3 = [3]_3$. We can therefore say that an equivalance relation on a set X imposes a partition on X. (eg. equivalence relation "has same colour" partitions the set of all cars into disjoint sets containing cars of specific colours. ie. xRy iff x & y share some property)

Union operation ← takes care of this.

n - Partitions

↑



NOTE: $[0]_n = [n]_n$

← $\mathbb{Z}/_{n\mathbb{Z}}$. It is the set consisting of all these disjoint subsets.

We have

$[x_i]$, $[x_i]_n \in \mathbb{Z}/_{n\mathbb{Z}}$

$$\bigcup_{i=0}^{n-1} [x_i]_n = \mathbb{Z}$$ ← We have recovered $\mathbb{Z}$

$$[x_i] \cap [x_j] = \emptyset$$ ← The Subsets of $\mathbb{Z}/_{n\mathbb{Z}}$ are disjoint. (Unequal) provided $[x_i] \neq [x_j]$ eg $[0]_3$ & $[3]_3$ $= [5]_3$

Else $[0]_3 \cap [3]_3 = [0]_3$ ~~[3]~~ $[3]_3$

ALSO NOTE: $[x_i]$ & $[x_j]$ are classes of $\mathbb{Z}$

We define arithmetics on $\mathbb{Z}/_{n\mathbb{Z}}$ as follows

$[a]_n + [b]_n = [a+b]_n$

$[a]_n - [b]_n = [a-b]_n$

$[a]_n [b]_n = [ab]_n$

In this way $\mathbb{Z}/_{n\mathbb{Z}}$ becomes a commutative ring.

eg. In the ring $\mathbb{Z}/_{24\mathbb{Z}}$: $[12]_{24} + [21]_{24} = [33]_{24} = [9]_{24}$ ← Arithmetic for 24-hour clock.

When $n=0$, $\mathbb{Z}/_{n\mathbb{Z}}$ does not have zero elements, rather, it is isomorphic to $\mathbb{Z}$ since $[a]_0 = \{a\}$

eg. $[1]_0 = \{\dots, 1-2(0), 1-1(0), 1, 1+1(0), 1+2(0), \dots\} = \{1,1,1,\dots\}$

$\{1,1,1\} \to 1$

$\{2,2,2\} \to 2$

$\{n,n,n\} \to n$

For $[x_i] \in \mathbb{Z}/_{0\mathbb{Z}}$

$$\bigcup_{i=0}^{n-1} [x_i]_0 \text{ or } \left[\bigcup_{i=1}^{n} [x_i]_0 = \{\dots1,1,1,\dots\} \cup \{\dots,2,2,2,\dots\} \cup \dots \{n,n,n,\dots\}\right.$$

$\{\dots,0,0,0,\dots\}$ - - - - - - $\dots \cup \{\dots, n-1, n-1, n-1, \dots\}$

$\to \mathbb{Z}/_{0\mathbb{Z}} \cong \mathbb{Z}$

What if 2 equivalance relation are the same?
eg. On set of triangles eg. "is similar to" & "is same as"
The disjoint sets of R_1 & R_2 will be similar.

Theorem

- Every equivalance relation partitions X into equivalance classes.
Conversely, every partition arises ~~into~~ from an equivalance Relation.

Proof

Let X be a set and R be an equivalance relation.

We need to show ① $\bigcup_i [x_i] = X$

$X/_\sim = \{x_i\}$ where $i = 1,2,3,\ldots,n$ (n-classes) $\{x_1, x_2, x_3, \ldots x_n\}$

② Either $[x_i] \cap [x_j] = \emptyset$
or $[x_i] = [x_j]$

① $\bigcup_i [x_i] \subseteq X$ Union of all or some equivalence classes of X is always a subset of X.

If $x \in X$ then $x \in [x] \longleftarrow = \{y \mid yRx\}$ $y \in X$ xRx
$\hookrightarrow$ x is always present in its own equivalence class as the reflexive property of the equivalance relation must hold. *

So $x \in \bigcup_i [x_i]$ and $\subseteq X$
(iii)

Union of all the classes

i is an index → $[x_i]$ is an index for Union. *
Note the union ~~consists of~~ contains of the class $[x]$

some classes of X

non-Empty Intersection.

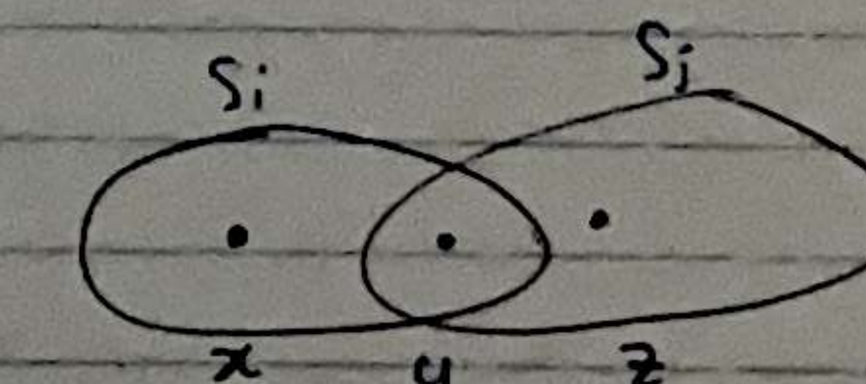
② If $z \in [x_i] \cap [x_j]$ then $z \in [x_i]$ and $z \in [x_j]$
So we have zRx_i and zRx_j * (x is always present in its own eq. class)
By symmetric property we have x_iRz
By transitivity x_iRz, $zRx_j \rightarrow x_iRx_j$
So $x_i \in [x_j]$
⇛ If for some arbitrary $w \in [x_i]$ then wRx_i and wRx_j (as $x_i \in [x_j]$)
So $[x_i] \subseteq [x_j]$
In a similar way, it can be shown $[x_j] \subseteq [x_i]$
$\Rightarrow [x_i] = [x_j]$

Relabeling Rule ⟵ Review This!! 6 is present in both.

If $y \in [x]$ then $[x] = [y]$ (As in the $[0]_3$ & $[3]_3$ examples)
$3 \in [0]_3 \Rightarrow [3]_3 = [0]_3 \leftarrow (= [6]_3)$

Conversely Suppose ~~$[x_i]$ is a~~ $\{S_i\}$ is a partition of X where S_i is the name of the set of the equivalance class $[x_i]$
We need to get an eq. rel. out of this partition.
~~Defn~~ of eq. relation: xRy if x & y belong to same partition/class for this case S_i

Show {
Ref. xRx
Sym. $xRy \rightarrow yRx$
~~Transitiv~~
Tra. $xRy, yRz \rightarrow xRz$ →



$xRy \rightarrow x$ & y in S_i
$yRz \rightarrow y$ & z is S_j ← z was defined in S_j
$y \in S_i \cap S_j \rightarrow S_i = S_j$ so x & z in S_i ← z is now in S_i
↑ non Empty intersection $\Rightarrow xRz$

Note: Different equivalance relations create different partitions.

$$X = \{\overset{\text{Red}}{R}\underset{\text{year}}{68}, R95, \overset{\text{blue}}{B}68\}$$

R_1 = Same year

Then

$$X = \{R68, B68\} \cup \{R95\}$$

R_2 = Same Colour

$$X = \{R68, R95\} \cup \{B68\}$$

Partitions

Composition of Relations ??

Partial Order relation.

- In Order Theory, a partially ordered set (poset) formalises and generalises the intuitive concept of an ordering, sequencing or arrangement of the elements of a set.

~~- A (non-strict) partial Order over a Set P is a binary relation over a set P~~

Def$\underline{\underline{n}}$: A (non-strict) partial Order is a binary relation ($\preceq$) over a set P Satisfying the axioms of ① Reflexivity ② Antisymmetry ③ Transitivity.

- In a poset P, consider elements a and b. If $a \preceq b$ or $b \preceq a$ then a and b are said to be comparable. That is the partial order relation exists between a & b.

In the case of a total-order relation over a set T, all the elements of T are comparable.

Orders & Lattice

Partial Order and Hasse Diagrams T

Motivation - Suppose we want a relation "is taller than"

$a T b \Rightarrow b T a$ X a is taller than b does not imply b is taller than a.

$\Rightarrow$ Taller relation cannot be classified as an equivalance Relation.

This leads to a concept called "Ordering" where relation holds from x to y and not from y to x.

Partial Order $(P, \overset{\preceq}{\leqslant})$

Where both are assumed True.

- Reflexive $\forall x \; xRx$
- Antisymmetric $\forall x \forall y \; xRy \neq yRx$ ie. $(xRy \wedge yRx) \rightarrow x = y$
- Transitive $\forall x \forall y \forall z \; xRy \; yRz \rightarrow xRz$

NOTE : $(P, \leqslant)$

↳ $<$ - Hilights that relation is unidirectional

↳ Hilights that relation is (~~transitive~~ reflexive)($=$ symbol)

Ex 1) People $= \{a, b, c\}$

Relation (P) : "Not richer than"

We have : aPb - a is not richer than b.

aPc a is not richer than c.

Does the relation p form a partial order?

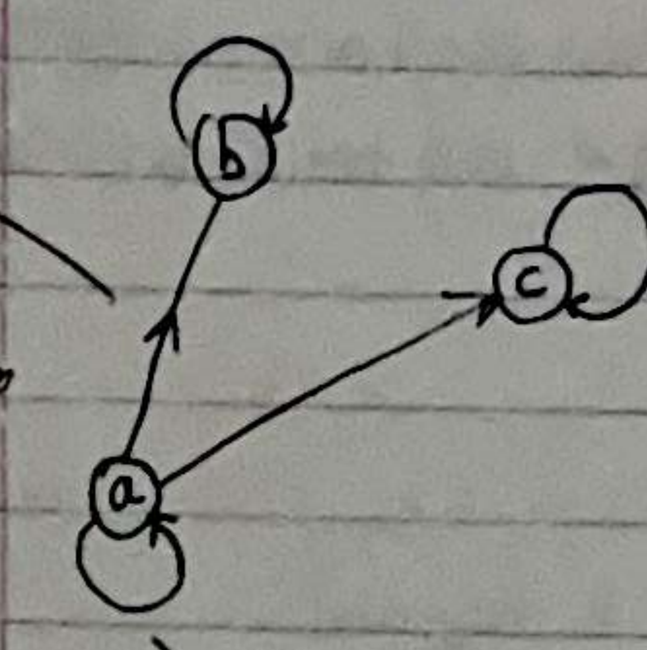
a is not richer than a $\rightarrow$ True $\Rightarrow$ Reflexive.

$aPb \Rightarrow bPa \rightarrow$ False $\rightarrow$ b has to be richer than a $\Rightarrow$ Antisymmetric

$aPb \; bPd \Rightarrow aPd \rightarrow$ True $\rightarrow$ Transitive

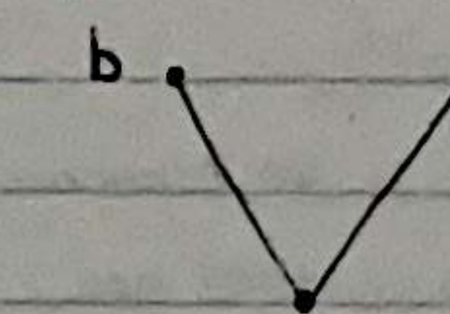
$\Rightarrow$ P is a partial Order Relation.

Graph for the partial order P



We note : Every node has a self-loop and graph is unidirectional

- We can Eliminate the Self-loops and remove the arrows and assume that relation holds from lower Element to upper.



— Hasse Diagram.

(a has lower order and b, c higher order as relation is from a to b, a to c)

Convention : Relation is from bottom to top.

Ex2) $P = \{3, 6, 18, 24, 72\}$

$(P, \leqslant)^{\preceq}$

↳ a factor of b

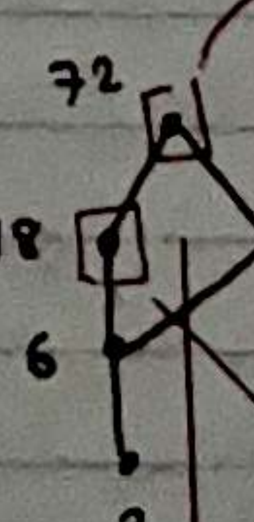
Draw the hasse diagram.

$3 \overset{?}{\leq} 72$, $6 \overset{?}{\leq} 72$, $18 \overset{?}{\leq} 72$

, Satisfy the condition "factor"

All 3, 6 & 18 are factors of 72 (comparable)

In a Hasse Diagram - Any node which are at a lower level to a certain node and also connected to that node will satisfy the property.



There we have 2 Elements as base elements So we need to check whether both 18 & 24 are factors of 72

We note that 18 is not a factor of 24, so we check with the previous Element ie. 6

24 is a factor of 6 so we put it above 6

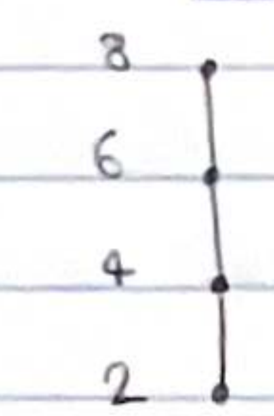
For 18 only 6 & 3 satisfy the condition "factor"

Elements at same level and above do not satisfy.

$18 \not\leq 24$

Note: A ~~po~~ set $(P, \preceq)$ is said to
*be well ordered if ① P is finite
② $\preceq$ is a total-order on P.

EX3) $P = \{2, 4, 6, 8\}$
$(P, \preceq)$ $<$
a less than b
(strict ??)

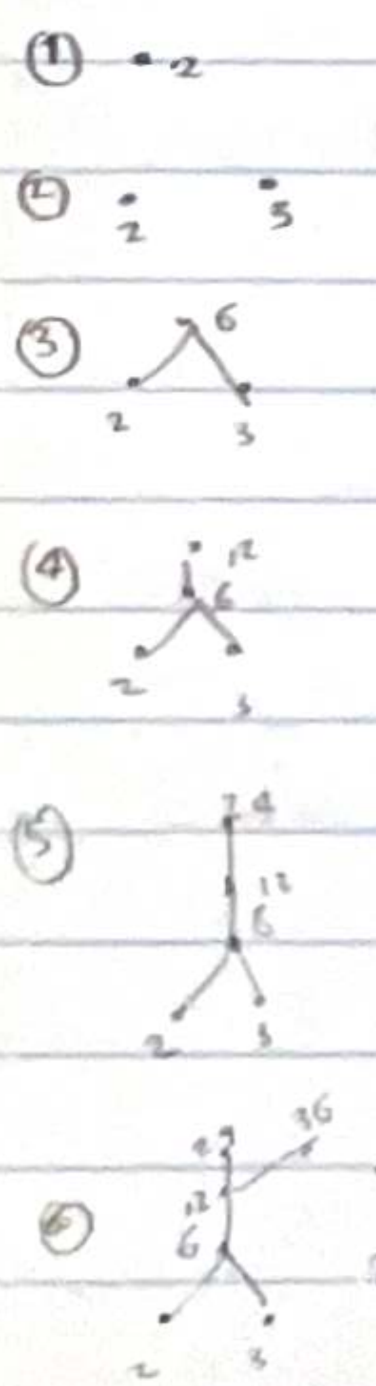


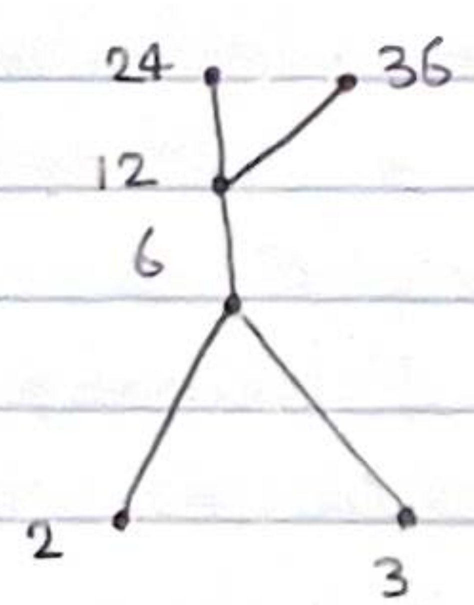
→ No branches (chain)
Also Any 2 elements can be compared
~~4~~ 2 is less than 4 (2,4)
4 is less than 6 (4,6)
2 is less than 8 (2,8)
→ This is called Total Order.

→ Not total order as these 2 elements are not comparable.

$\preceq$ be such that if

EX4) Let $X = \{2, 3, 6, 12, 24, 36\}$ and the relation $x \preceq y$ if x divides y.
Draw the Hasse diagram of $\langle X, \preceq \rangle$





24 does no divide 36 but 6 does.

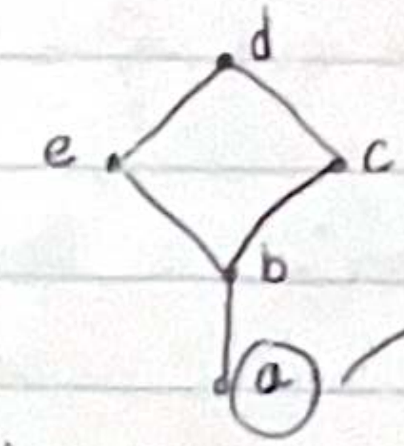
Well-Ordering Principle *
- Every non-empty subset of $\mathbb{N}$ has a smallest (or first) element.

The Well ordering Principle also holds for $\mathbb{N}_{\neq 0}$

Proof → Need to do this.
Set difference

Least, Greatest, Minimal and Maximal Elements

Suppose we have a partial Order P $\langle P, \preceq \rangle$ – Poset
$\langle P, \preceq \rangle - \{ y \in P \mid \forall x \in P, y \preceq x \}$ such that
↳ The element y is (less than or equal to) all the elements in P. ie. y is the least member in p.
in the set P
smallest according to the partial order relation.
→ $a \preceq b$ ⇒ a precedes b (behaves similarly to $\leq$)

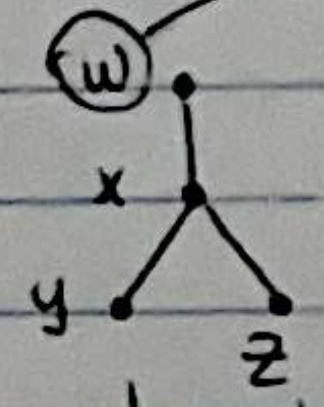


a is the least element in P.
nothing precedes a

↳ This diagram can be represented as follows. $\langle P, \preceq \rangle = \langle \{a, b, c, d, e\}, \preceq \rangle$
$a \preceq b$
$b \preceq c$
$b \preceq e$
$e \preceq d$
$c \preceq d$
Use transitivity to get others
$a \preceq b$ $b \preceq c$ $c \preceq d$ ⇒ $a \preceq d$
$a \preceq b$ $b \preceq e$ $e \preceq d$ ⇒ $a \preceq d$
a is the least Element in all cases.

$\{ y \in P \mid \forall x \in P, y \succeq x \}$ (or $x \preceq y$) → As seen we can similarly conclude that d is the greatest element in P.

- Often, least member is represented as 0 and greatest member as 1.
ie a is 0, d is 1

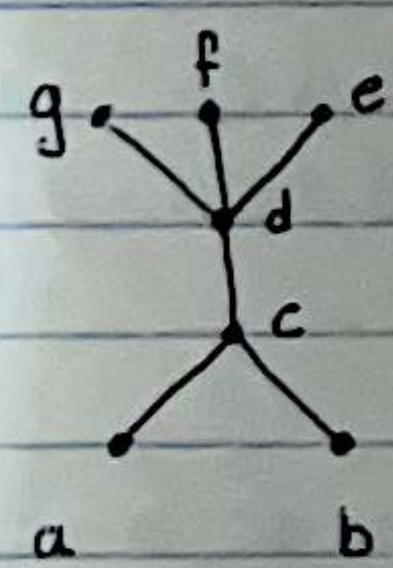
Ex2) 

Greatest Element

y and z at the same level ie $y \not\leq\not\geq z$ (non-comparable) → Partial Order relation cannot compare y & z to find the smallest one.

- The least element is one (unique) such that it is the smallest one in the whole set. (Definition)
- We see that there is no least member

NOTE : Greatest or Least Elements are unique if they exist.

Ex3) 

We can clearly see that there are no least/greatest member. (Non-Unique)

To address this problem we define a new class called:

minimal member

$(P, \leq) - \{ y \in P \mid \text{No } x \in P, x \not< y \}$

→ Set of elements of P for which there exist no elements smaller than the ones in the set. (Can be at same level AND NO elements lower than them)

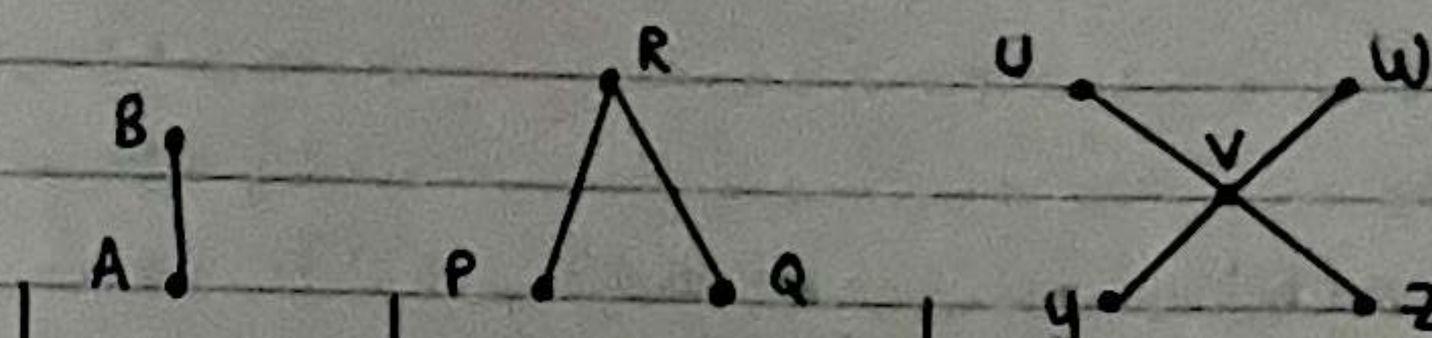
minimal members $= \{a, b\}$

maximal members $= \{ y \in P \mid \nexists x \in P, y < x \}$ ← Biggest element.

$= \{e, f, g\}$

NOTE : Maximal/Minimal members need not be unique.
Distinct Min/Max members are incomparable (same level)

Ex 4)



Least	A	–	–
Greatest	B	R	–
Minimal	{A}	{P, Q}	{y, z}
Maximal	{B}	{R}	{U, W}

Lower bound, Upper bound, Least upper bound (infimum) and Greatest lower bound (Supremum)

Suppose $\langle Q, \leq \rangle \subseteq \langle P, \leq \rangle$ (Q is a subset of P) Both P & Q are Partial Orders.

$x \in Q$ $y \in P$

$\forall x \ y \leq x$

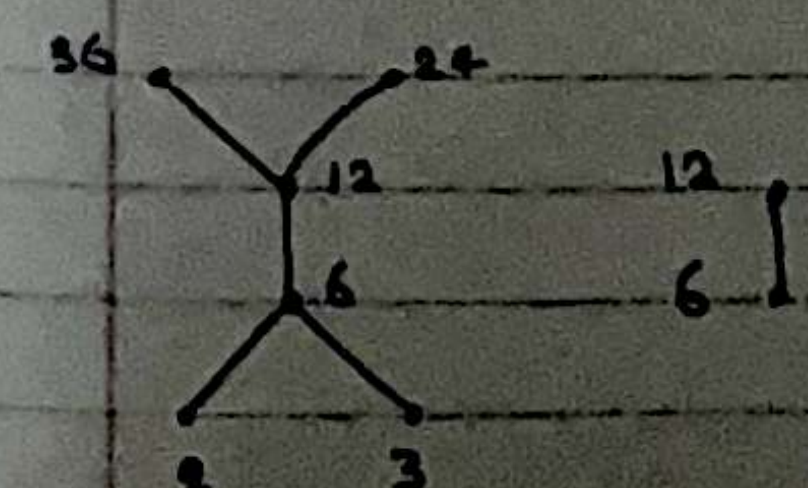
→ less or equal to (in some way $\preceq$ and $\leq$ are the still same) (a less than b and a precedes b)

y is called a lower bound. ~~Trivially the smallest element in the subset for ≤ and largest for ≥~~ any element belonging to the subset.

$\forall x \ x \leq y$ then y is called the upper bound. and the values should be less or equal to (entire) (we choose values of the larger set).

Ex1)

$\langle M, \leq \rangle$ $\langle N, \leq \rangle$, $N \subseteq M$



Lower bounds $= \{2, 3, 6\}$ NOTE: We do not include 12 as although $12 \leq 12$, $12 \not\leq 6$.

2 is both ≤6 and less than or equal to 12

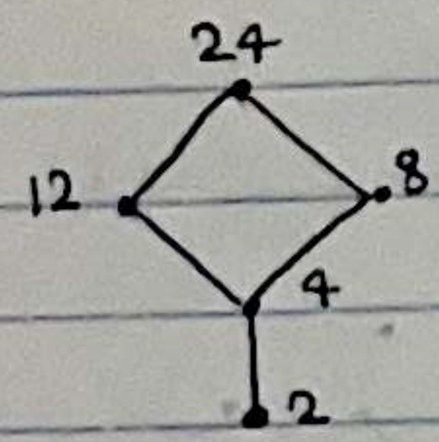
6 is both ≤6 and ≤12

Upper bounds $= \{12, 24, 36\}$

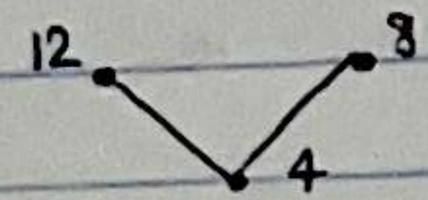
Check for both elements of subset $\begin{cases} 6 \leq 12, 24, 36, 6 \\ 12 \leq 12, 24, 36 \text{ but NOT } 6 \end{cases}$

Ex 2)

$\langle X, \leq \rangle$



$\langle Y, \leq \rangle$



The LB must be less or equal to all elements of the subset

Sub- In set

$LB = \{ y \mid \forall x \; y \leq x \}$

In set

$UB = \{ y \mid \forall x \; y \geq x \}$

The UB must be greater or equal to all the elements of the subset.

LB (Any thing below subset can be taken).
(Also, if the subset has a least element the latter is also taken)

$LB = \{2, 4\}$ [We do not take 8 as the elements must be less than (or equal to) ALL the elements in the subset and $8 \nleq 4$ ← least element in subset.]

UB (Anything above subset is taken and the greatest element of the subset is also taken.)

$UB = \{24\}$ We do not include 12 & 8 as $12 \neq 8$ (cannot be compared and hence cannot conclude which is further / greater).

If the greatest element (or least) in the subset (non-comparable subset) cannot be directly found — in case of LB — we stop searching.

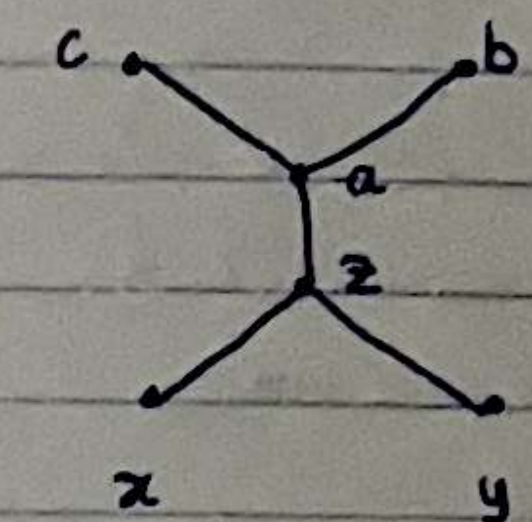
4 is not taken in UB Why??

~~The subset contains an element larger than 4.~~

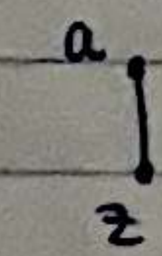
[We do not take 4 as the elements must be $\geq$ all the elements in the subset and $4 \ngeq$ 8 or 12]

Ex 3)

$\langle P, \leq \rangle$



$\langle Q, \leq \rangle$



$LB = \{x, y, z\}$

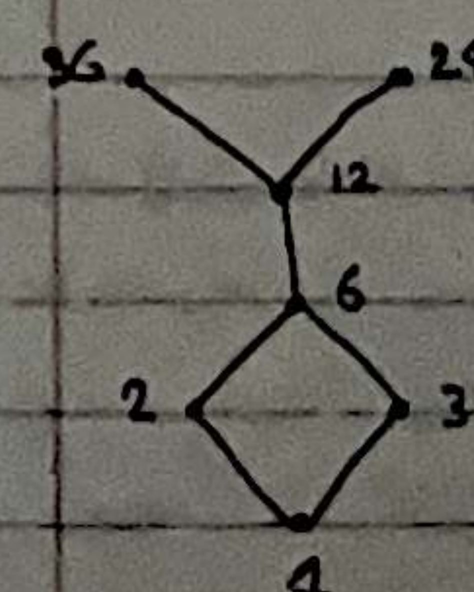
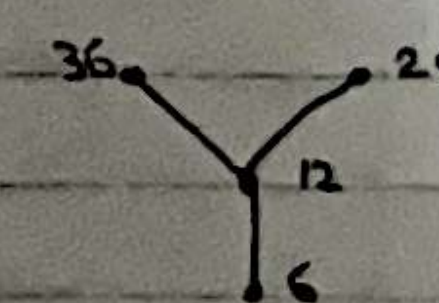
Out of the 3 lower bounds there is one which is closest to the (least element of) the subset.
ie. z is the greatest lower bound (GLB)
z is called infima. (infimum)

$UB = \{a, b, c\}$ → The "closest" element to the greatest element of the subset is a and is called the Least upper bound
a is the Suprema (Supremum)

NOTE : Infima and Suprema may or may not exist. If they do, they are unique.

Ex 4)

set — subsets

P	Q	R	S
			
	$LB = \{1, 2, 3, 6\}$	$LB = \{1\}$	$LB = \{1, 2, 3, 6\}$
	$UB = \{12, 24, 36\}$	$UB = \{6, 12, 24, 36\}$	$UB = \{\emptyset\}$
	Inf = 6	Inf = 1	Inf = 6
	Sup = 12	Sup = 6	Sup = No LUB

Lattice

- It is a partial order $(L, \leq)$
- $\forall (a,b) \in L$ For all pair of elements in L Each pair must have an infimum (GLB) and a Suppremum (LUB)

These conditions must be satisfied for L to be a lattice.

Representation

(LUB)

$Sup(a,b)$ is represented as $a \vee b$ - This is called Join $(a+b)$

$inf(a,b)$ is represented as $a \wedge b$ - This is called Meet. $(a \cdot b)$

(GLB)

NOTE: A partially ordered set $(L, \leq)$ is called a join-semilattice if each two-element subset $\{a,b\} \subset L$ has a join (i.e. least upper bound)

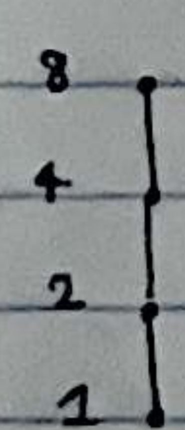
It is called a meet-semilattice if each 2-element subset $\{a,b\} \subset L$ has a meet (greatest lower bound)

A $(L, \leq)$ is called a lattice if it ~~has~~ is both a join- and meet-semilattice.

Ex1) Consider the following partially ordered set.

$S_8 = \{1,2,4,8\}$

factors of 8.



We shall check if S_8 is a lattice.

$\boxed{\forall (a,b) \in S_8 \ \exists \ inf(a,b) \wedge Sup(a,b) \Rightarrow (S_8, \wedge, \vee)}$ Succession!

Consider $(1,2)$

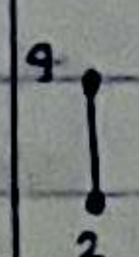
$inf(1,2) = 1$ Meet

$Sup(1,2) = 2$ Join

$(2,4)$

$inf(2,4) = 2$ meet

$Sup(2,4) = 4$ join



Elements pair need not be in ^

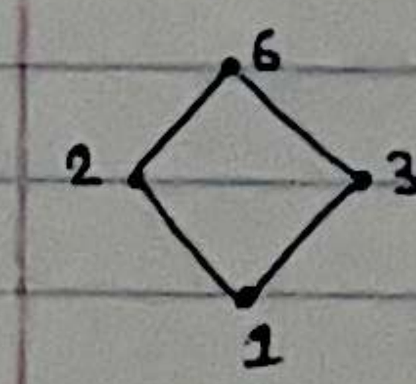
$(2,8)$

$inf(2,8) = 2$ Meet

$Sup(2,8) = 8$ join.

Ex2) $S_6 = \{1,2,3,6\}$ $x|y$

factors of 6.



Consider $(2,6)$

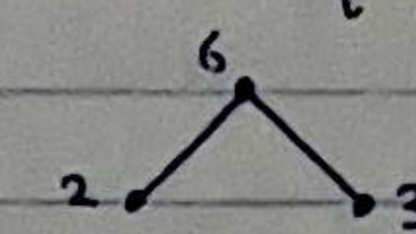
$LB = \{1,2\} \Rightarrow inf(2,6) = 2 \rightarrow$ Meet

$UB = \{6\}$ $Sup(2,6) = 6 \rightarrow$ Join.

Same argument for $(1,3)$

$(1,2)$

$(3,6)$



Consider $(2,3)$

$2 \not\rightarrow 3$

We cannot say 2 is l.b as it cannot be compared to 3

But we can see that 1* is the l.b for $(2,3)$ and 6* is the u.b for $(2,3)$. (Same as EX4 R)

No least/greatest element in the subset $\{2,3\}$ can be defined so we do not include them and we take elements below/above the subset.

$\Rightarrow LB = \{1\}$ $UB = \{6\}$

inf → Meet

Sup → Join.

Same for $(1,6)$

$\Rightarrow (S_6, \wedge, \vee)$

EX 3) $S_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

$(S_{24}, \leq)$ with $x \leq y \to x | y$ or $y | x$??

↳ x divides y

20.0

~~1.5 /semester~~

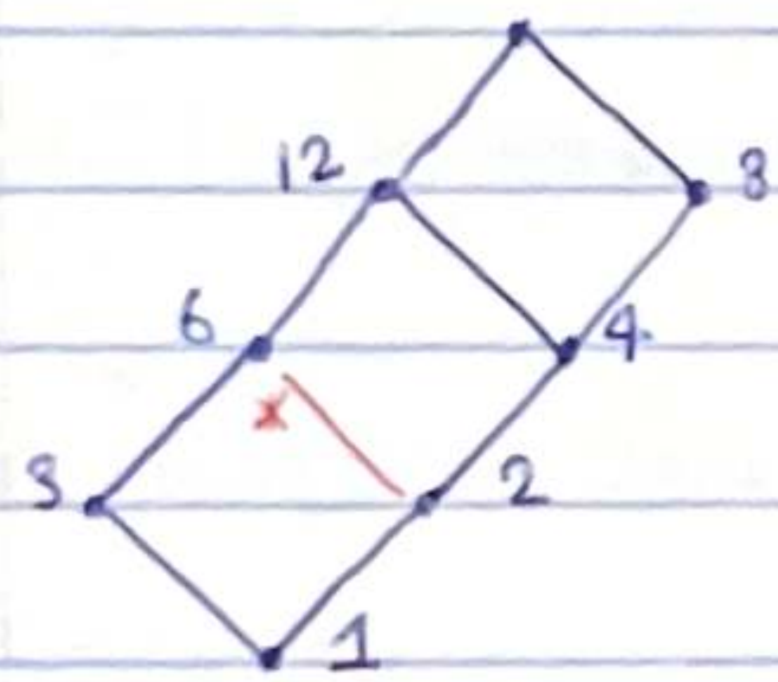
~~⇒ 3.0 / year~~

3.0 × 4 = 12.0

↳ Hasse Diagram of S_{24}

* both 2 & 3 can divide 6

⇒ We can join.



→ This forms a lattice.

Show : (1, 3)

No elements below 1

$\begin{cases} \inf(1,3) = 1 \quad \text{from } \{\emptyset, 1\} \\ \sup(1,3) = 3 \end{cases}$

↳ LUB

(1, 6)

inf(1, 6) = 2

Sup(1, 6) = 3

⇒ $(S_{24}, \wedge, \vee)$