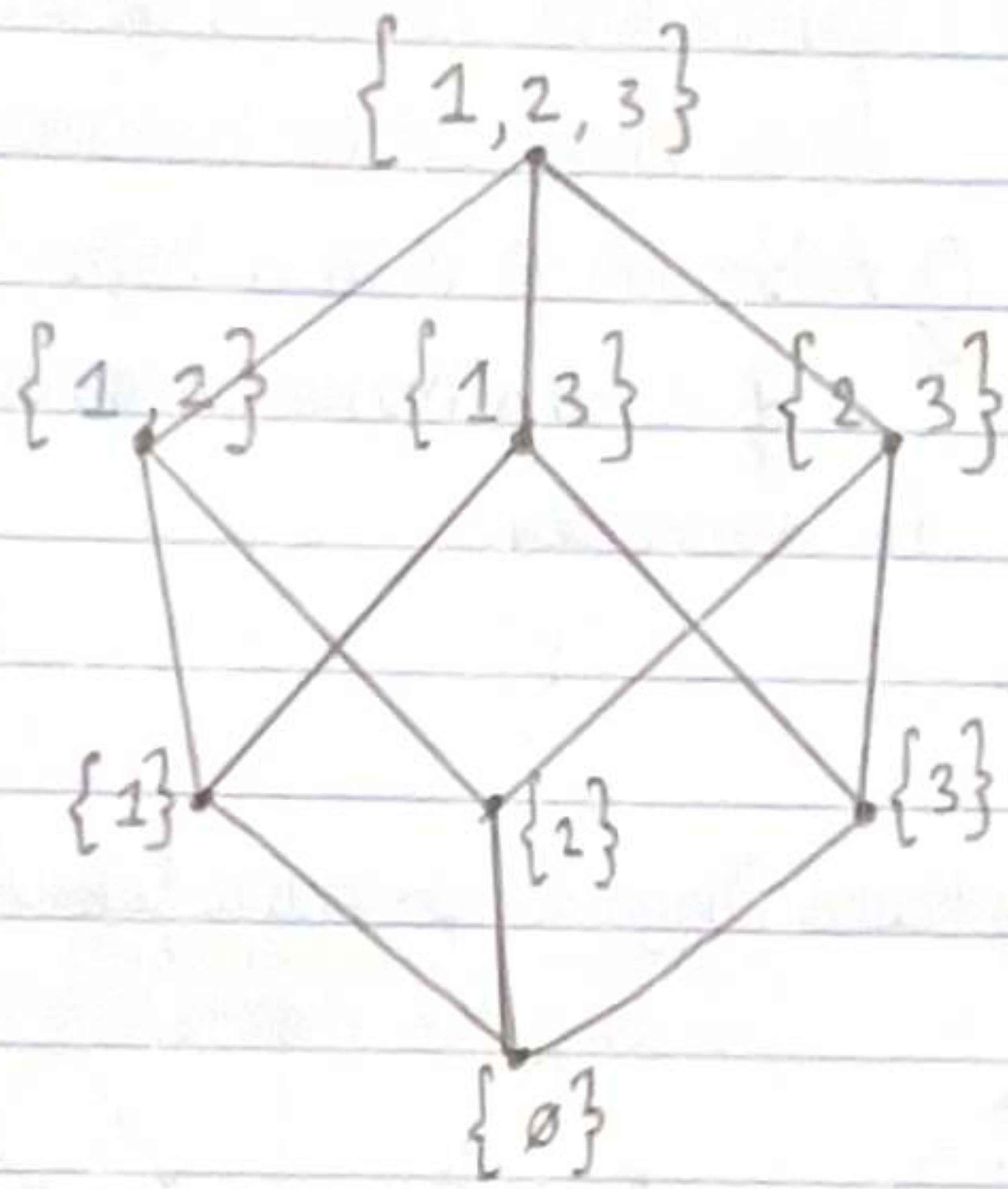
Ex4) $S=\{1,2,3\}$

or $S=\{x,y,z\}$

Lower Element is a subset of higher Element.

Draw the Hasse Diagram for $\langle \mathcal{P}(S), \subseteq \rangle$ ⟶ Poset of Power Set of S under inclusion.

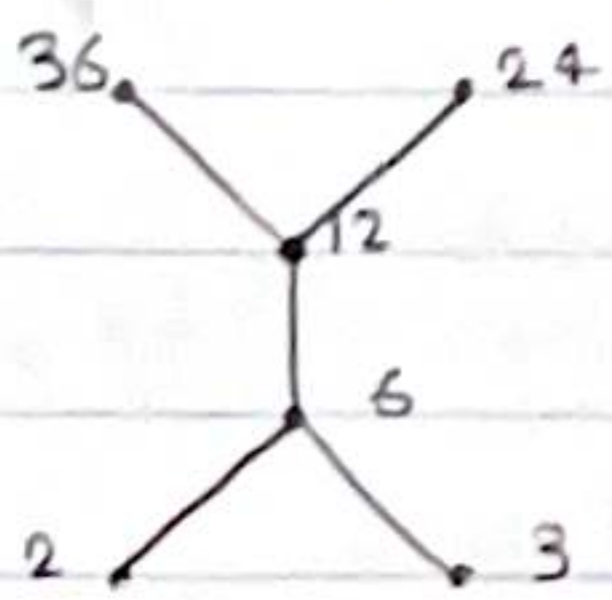
$$\mathcal{P}(S)=\{\{\emptyset\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$$



← NOTE: This is known as a distributive lattice.

Non-Example of Partial Orders which are NOT lattices.

1)



Consider (6, 12)

$LB=\{2,3,6\}$ → inf

$UB=\{12, 24, 36\}$ → Sup

Consider (24, 36) Join.

$UB=\{\emptyset\}$ ⟶ No Sup ⇒ No $\wedge$

$LB=\{12\}$

* Consider (3, 24)

$UB=\{6,12\}$??

$LB=\{\emptyset\}$ → No inf

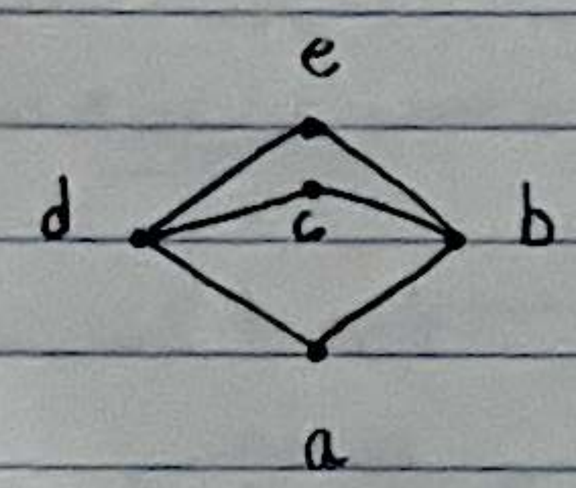
① Consider (2, 3)

$UB=\{6\}$

$LB=\{\emptyset\}$ → No infimum ⇒ No Meet

Reject lattice.

2)



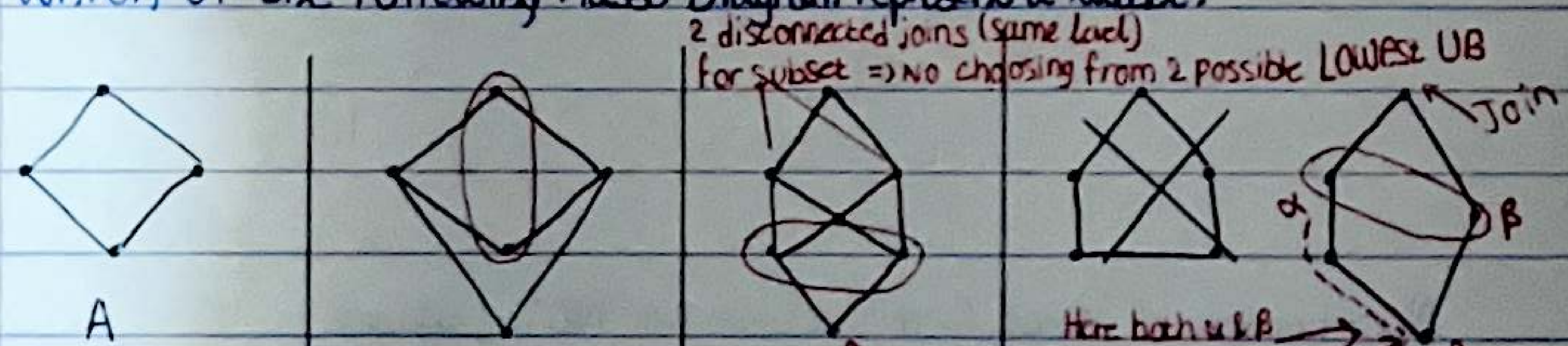
① We Start by looking at pair of elements with no relation connections eg. (C, e)

② LB (C, e) is d and b. But the 2 are at the same level hence b ≱ d and a Lower bound cannot be determined. (Cannot compare hence cannot know which is greater.) Therefore greatest lower bound cannot be known. ⇒ NO MEET.

③ Although e is at a higher level than c, c ≱ e and hence no lowest upper bound can be determined.

Cannot be compared
No connection.
⇒ No Lattice.

3) Which of the following Hasse Diagram represent a lattice?

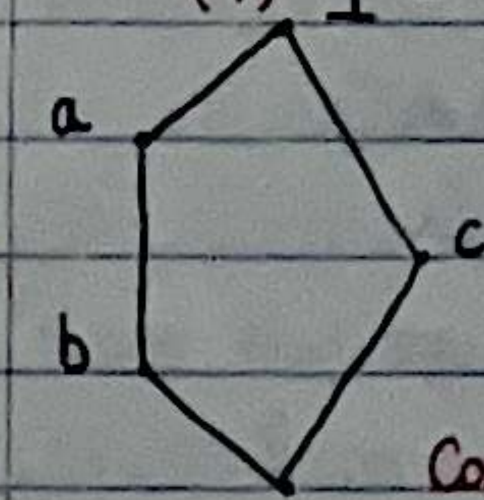


A	B	C	D
Lattice ✓✓ As shown Before	Condition for lattice fails for the selected Subset.	Meet ⇒ No Lattice.	Here both α & β are connected where (α, β) non-connected pair. ← LB for α & β ⇒ Lattice.

Lattice Algebra & Distributive Lattices

Consider the following lattice.

(T) 1 ← Join



0 (⊥) meet

$a \wedge a = a$ (a meet a = a) operation
$a \vee a = a$ (a join a = a) operation
} Idempotent laws

Commutative law { $a \wedge c = 0$ (a meet c means we look for the value inf(a,c))
{ $c \wedge a = 0$

$a \wedge c = c \wedge a$
$a \vee c = c \vee a$

GLB(a,b) = b = inf(a,b)

$(a \wedge b) \wedge c = b \wedge c = 0$
$a \wedge (b \wedge c) = a \wedge 0 = 0$
$\Rightarrow (a \wedge b) \wedge c = a \wedge (b \wedge c)$
$(a \vee b) \vee c = a \vee (b \vee c)$ } Associative laws

~~$a \wedge (a \vee c) = a \wedge (1) = a$~~
~~$a \vee (b \wedge c) = a \vee (0) =$~~ GLB from {a, 1} = a *

$a \wedge (a \vee c) = a \wedge (1) = a$
$a \vee (a \wedge c) = a \vee (0) = a$
} Absorption Laws

↘ lowest upper bound from {a, 1} * is a

* Review This.

IDENTITY LAWS

$a \vee 0 = a$
$a \wedge 1 = a$
} ↖ from Absorption laws

Distributive

Check if ①, ②

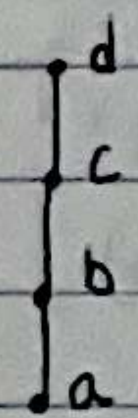
$a \wedge (b \vee c) = a \wedge b \vee a \wedge c$
$= (a \wedge b) \vee (a \wedge c)$
$b \vee 0 = b$ — ②

$a \wedge (b \vee c) = a \wedge (1) = a$ ①

① ≠ ②
⇒ Not Distributive

or $a \wedge (b \vee c) > a \wedge b \vee a \wedge c$
↘ a is at a higher level than b

Consider the following lattice.



$$a \wedge (b \vee c) \overset{!}{=} a \wedge b \vee a \wedge c$$

$$\downarrow \qquad\qquad a \vee a$$

$$a \wedge (c) \qquad\qquad \downarrow$$

$$a = a$$

$\Rightarrow$ Distributive Lattice.

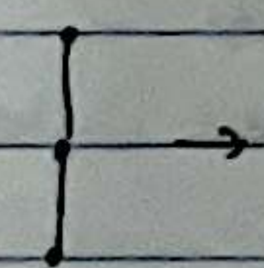
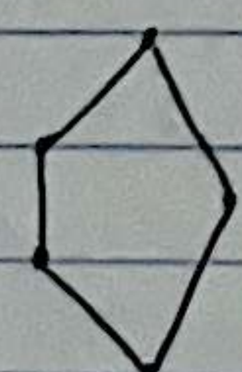
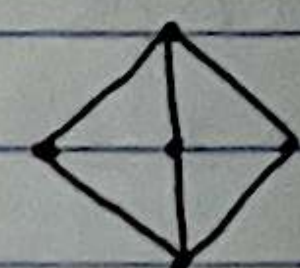
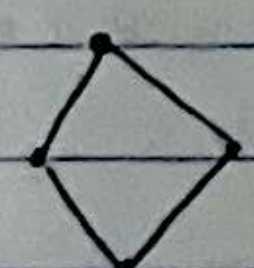
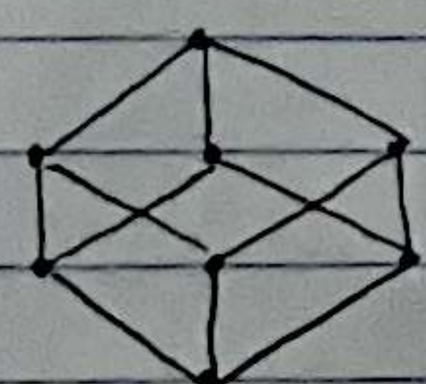
Note : Some lattices may or may not be distributive.

$\swarrow$ 2 possibilities greater or equal

$a \wedge (b \vee c) \geq a \wedge b \vee a \wedge c$) can also be written

$a \vee (b \wedge c) \leq a \vee b \wedge a \vee c$

$\nwarrow$ 2 possibilities less than or equal.

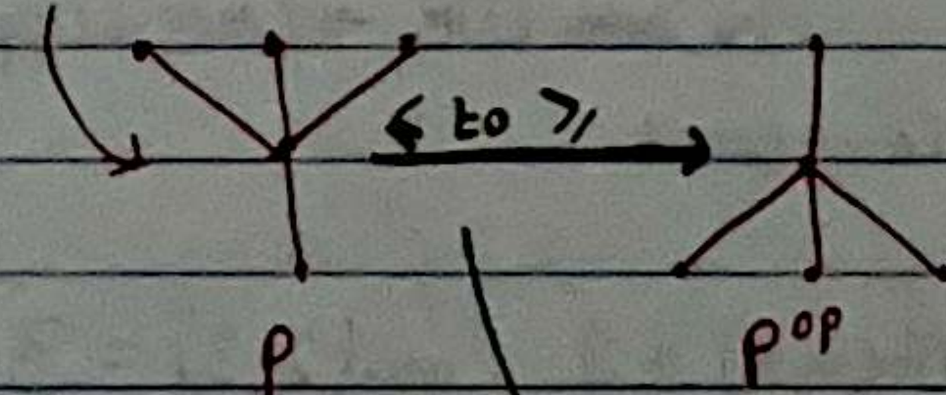
Examples of Distributive Lattices	Non-Distributive Lattice.
 → Chains	 
 → Distributive Too.	
 → Power set ordered under inclusion	

An algebraic Structure $(L, \vee, \wedge)$, consisting of a set L and two binary operations $\vee$ and $\wedge$ on L is a lattice if the previously shown axiomatic identities hold for all element a, b, c of L.

The axioms assert that both $(L, \vee)$ and $(L, \wedge)$ are semilattices. The absorption laws, the only axioms in which both meet and join appear distinguish a lattice from an arbitrary pair of semilattices structures and assure that the two semilattices interact appropriately. In particular, each semilattice is the dual of the other.

$\searrow$ In order theory, every partially ordered set P give rise to a dual (or opposite) partially ordered set which is often denoted P^{op}. It is the same set but with inverse order

$\searrow$ ① Their Hasse diagram are upside down ② $\geq \leq$ becomes $\geq$



$\searrow$ Note $\geq$ must be defined again.

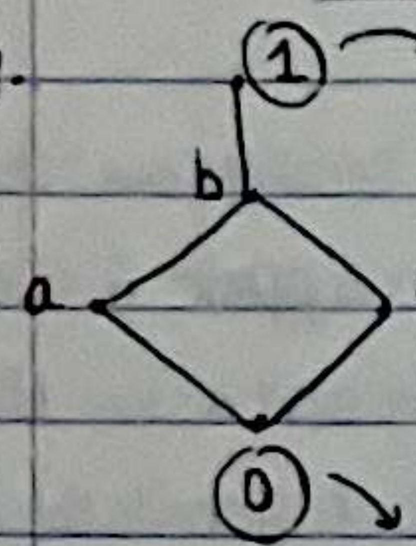
Bounded, Complete and Compliment Lattice

Bounded Lattice

Note : This is NOT reffering to a subset of L which has inf & Sup but the lowest & greatest Element of the entire lattice itself.

- It is a lattice which has a least and a greatest element.

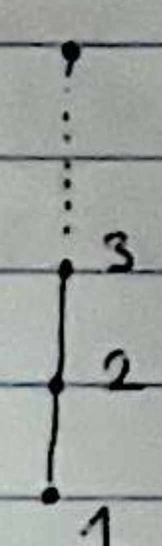
eg. (1) → Greatest → By defn of lattice all (x,y) have inf and Sup already.



⇒ Bounded Lattice.

(0) → least

Consider $\langle \mathbb{N}, \leqslant \rangle$



Does this form a lattice?

① Consider $(1,2)$

1 is inf 2 is Sup → ~~Sup~~ LUB → Lowest Upper Bound

GLB → Greatest Lower Bound

LUB

GLB

② Consider $(2,3) \rightarrow \inf(2,3) = 2 \quad \sup(2,3) = 3$

③ Consider $(2,4) \rightarrow \inf(2,4) = 2 \quad \sup(2,4) = 4$

$LB = \{1, 2\} \quad UB = \{4, 5, 6, \dots\}$

⇒ $\langle \mathbb{N}, \leqslant \rangle$ is a lattice.

ie. LUB & GLB Exist for all pairs

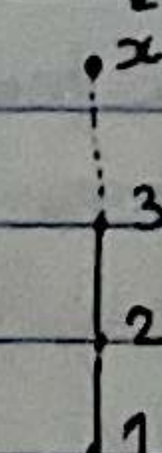
But is this a bounded Lattice?

Least Element $= 1$

Greatest Element = __ ?? No Greatest, No upper bound (on Entire lattice)

However, We can make $\langle \mathbb{N}, \leqslant \rangle$ bounded as follows:

We could add an element x, $\langle \mathbb{N}, \leqslant \rangle \cup \{x\}$ Such that $\forall a \in \mathbb{N}, a \leqslant x$

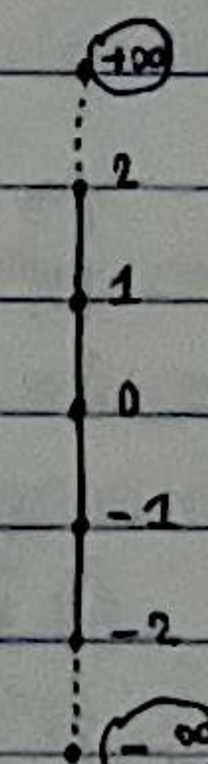


Now $\langle \mathbb{N}, \leqslant \rangle$ is a bounded lattice

with least element $= 1$ and Greatest element $= x$.

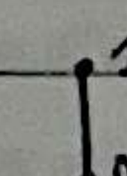
→ Suppose we have the following Lattice $\langle \mathbb{Z}, \leqslant \rangle$. We need to Complete it by adding $+\infty$ and $-\infty$ as Greatest and least elements respectively.

ie. $\langle \mathbb{Z}, \leqslant \rangle \cup \{+\infty, -\infty\}$



For adding upper bound ← LUB or Sup

Consider $(0,1)$



$0 \vee 1 = 1$

$1 \vee 2 = 2 \rightarrow 0 \vee 1 \vee 2 \vee 3 \vee \dots$

$2 \vee 3 = 3$

$\Rightarrow \bigvee \mathbb{Z} = 0 \vee 1 \vee 2 \vee 3 \vee \dots$

$= +\infty.$

Negative integers can also be included

ie. $-2 \vee -1 = -1 \quad -1 \vee 0 = 0 \quad 0 \vee 1 = 1 \dots$

Similarly for Lower bound

~~$2 \wedge 1 =$~~

$2 \wedge 1 = 1 \quad 1 \wedge 0 = 0 \quad 0 \wedge -1 = -1 \dots \rightarrow 2 \wedge 1 \wedge 0 \wedge -1 \dots$

$2 \wedge 1 = 1 \wedge 2$

~~$2 \wedge 1 \wedge 2 \wedge 1$~~

$\Rightarrow \bigwedge \mathbb{Z} = -\infty \qquad$ Where $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

Complete Lattice

- A complete lattice is one where each subset we pick up from the lattice has GLB (inf) and LUB (Sup).

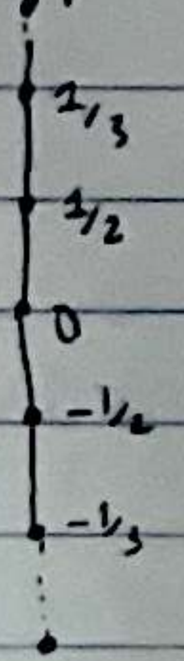
Now, we defined lattice as for all pairs (a, b), the latter has both inf and Sup. If it is satisfied for every 2-element then it is satisfied for every subset.

↳ Associativity of meet & join operations. ~~(Axiom)~~. (Laws) → 3, 4, 5, ... elements

NON-EXAMPLE.

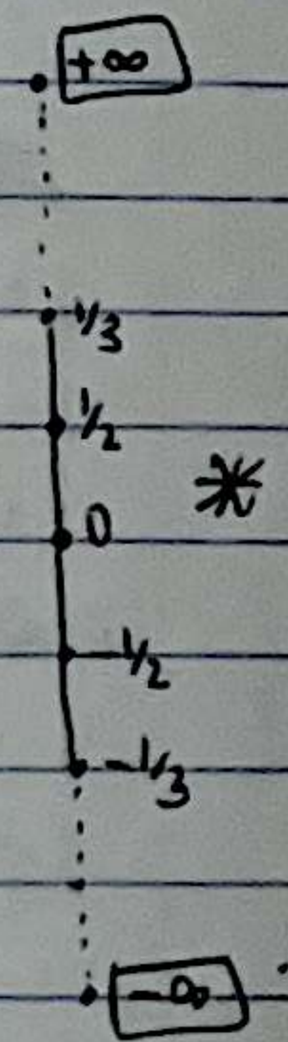
Suppose we have $\langle Q, \leq \rangle$

infinite ↑



Let's ~~bound~~ make $\langle Q, \leq \rangle$ bounded

$\langle Q, \leq \rangle \cup \{-\infty, +\infty\}$.



*

It is bounded, but is it complete?

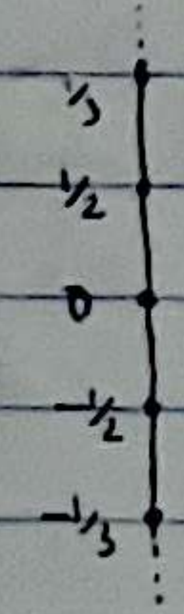
NO. From defn of complete lattice Each subset has a GLB and LUB.

Consider the subset M such that

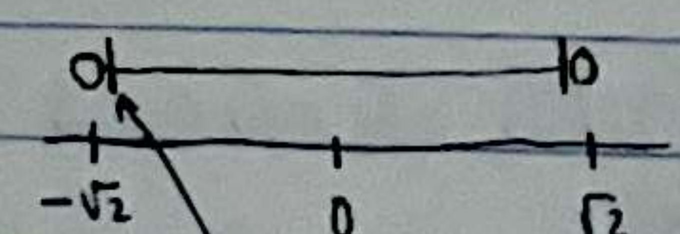
$M = \{a \in Q \mid a^2 < 2\}$

$\Rightarrow -\sqrt{2} < a < \sqrt{2}$

*



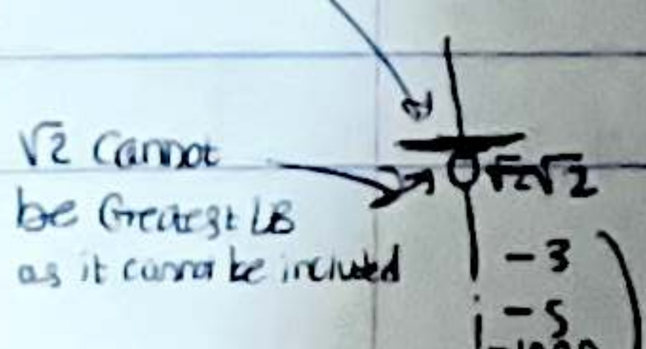
↳ What is the LUB & GLB for M



What is the number JUST HERE

↳ Rational ↳ The Greatest Lower bound (inf)

Just Here



$\cup\{-\infty, \infty\}$

- Although $\langle Q, \leq \rangle$ is bounded, the subsets do not have LUB & GLB hence $\langle Q, \leq \rangle \cup \{-\infty, \infty\}$ is bounded but NOT Complete.

↳ ~~This~~ $\boxed{\text{Infinite Sets may not become complete lattices.}}$

Finite Lattice ⇒ Complete

↳ We know all elements

and since it is a lattice each pair of elements have inf & Sup. This means that all possible subsets of a finite lattice have inf & Sup too. ⇒ Complete.

The inclusion of which of the following sets into $S = \{\{1,2\}, \{1,2,3\}, \{1,3,5\}, \{1,2,4\}, \{1,2,3,4,5\}\}$ is necessary and sufficient to make a complete lattice under the partial order defined on containment.

↳ Least no of element to satisfy completeness. (Just)

A $\{1\}$

B $\{1\}, \{2,3\}$

C $\{1\}, \{1,3\}$

D $\{1\}, \{1,3\}, \{1,2,3,4\}, \{1,2,3,5\}$

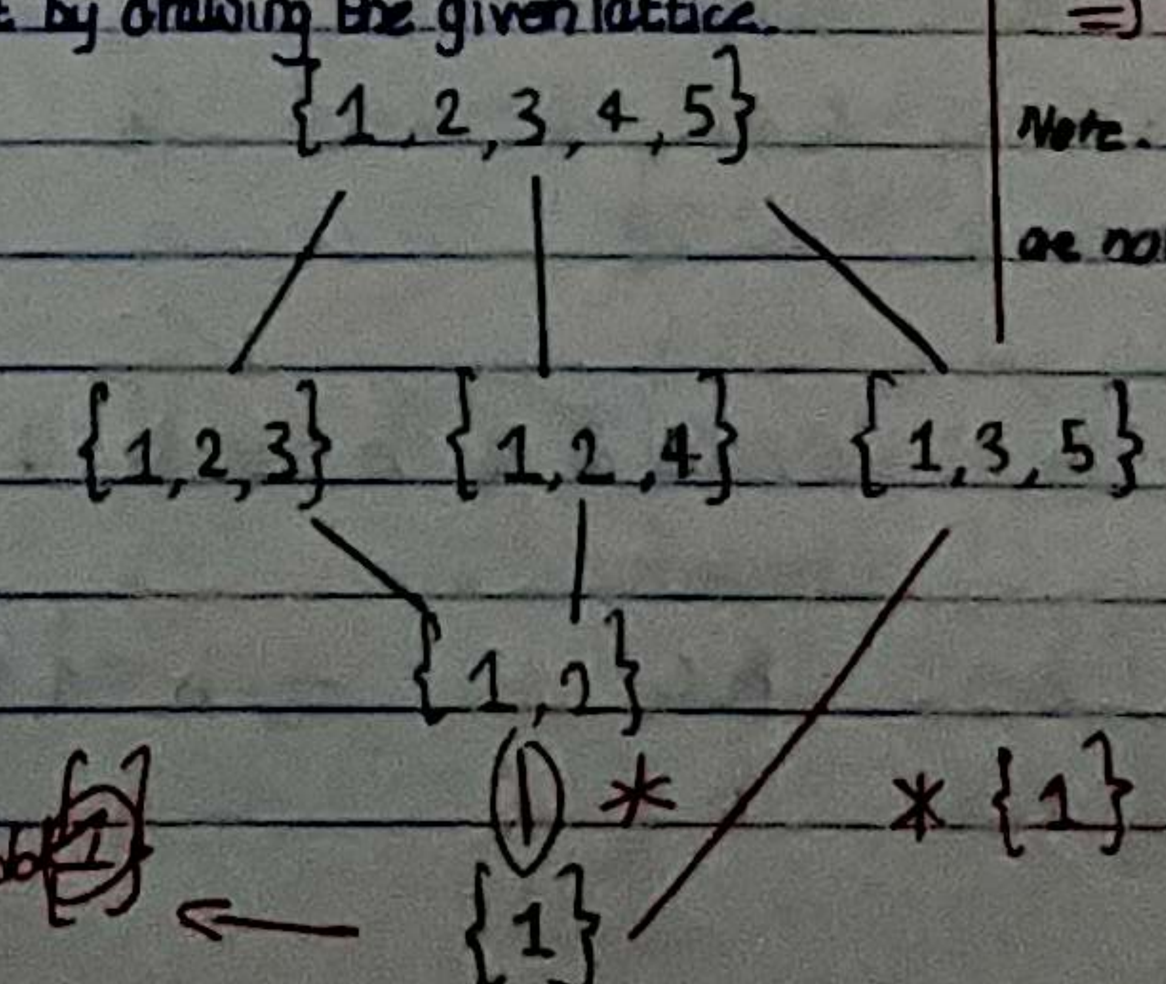
(N & S) Minimal Soln

③ We can see that by including $\{1\}$, the structure is finite and it can be shown that S is a lattice i.e. Each pair has inf and Sup

⇒ Complete Lattice.

Note. if Necessary & sufficient conditions are not mentioned, other answers are possible

① We start by drawing the given lattice.



They are at same level as they are incomparable i.e. none is the subset of the other.

but $\{1,2\} \not\subset \{1,3,5\}$

② We add $\{1\}$

* $\{1\} \subset \{1,2\}$ and $\{1,3,5\}$

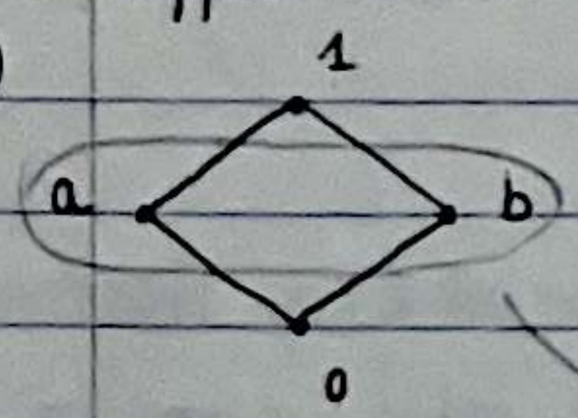
Complement Lattice

Suppose $a,b \in (L, \leq)$

a is complement of b if $a \wedge b = 0$ and $a \vee b = 1$ (least element: 0; greatest element: 1)

also b is complement of a.

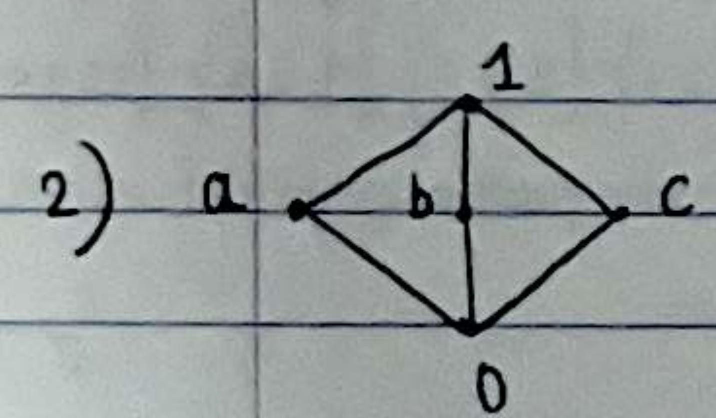
Suppose.

1) 

a' : which element in L where meet/join with a we get $0/1$.

Ans: b $\Rightarrow a' = b$ and $b' = a$

$a \wedge b = 0$ $a \vee b = 1$

2) 

a' ? TEST → Consider (a,b) | Consider (a,c)

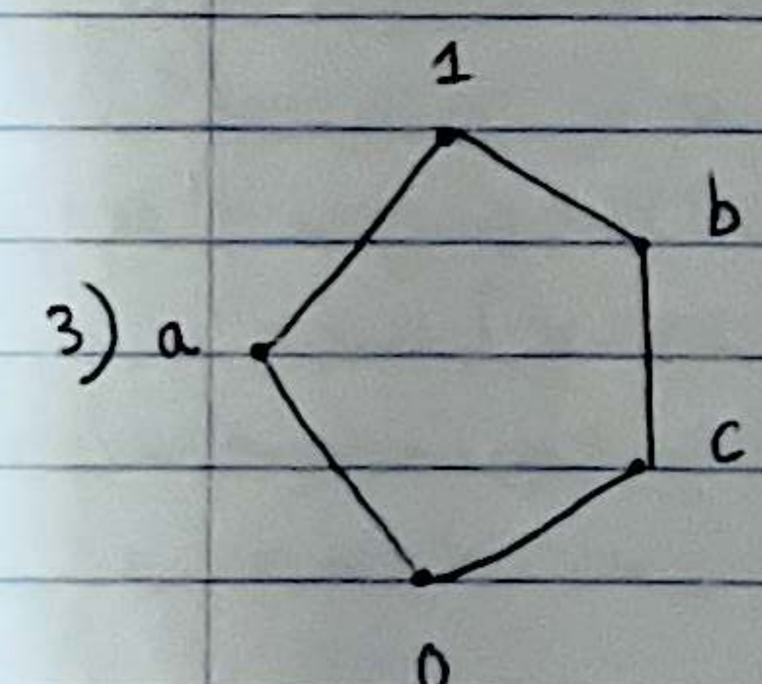
$a \wedge b = 0$ | $a \wedge c = 0$

$a \vee b = 1$ | $a \vee c = 1$

$\Rightarrow a' = b$ | $\Rightarrow a' = c$

$\Longrightarrow a' = \{b, c\}$

We can see that $b' = \{a, c\}$ and $c' = \{a, b\}$

3) 

a' : $a \wedge b = 0$ $a \vee b = 1$

$a \wedge c = 0$ $a \vee c = 1$

$\Rightarrow a' = \{b, c\}$

b' : We know if $a' = b$ then $b' = a$ $\Rightarrow$ We need to check for c.

$\begin{cases} b \wedge c = \cancel{b}\, c \\ b \vee c = b \end{cases}$ $\Longrightarrow b' = \{a\}$

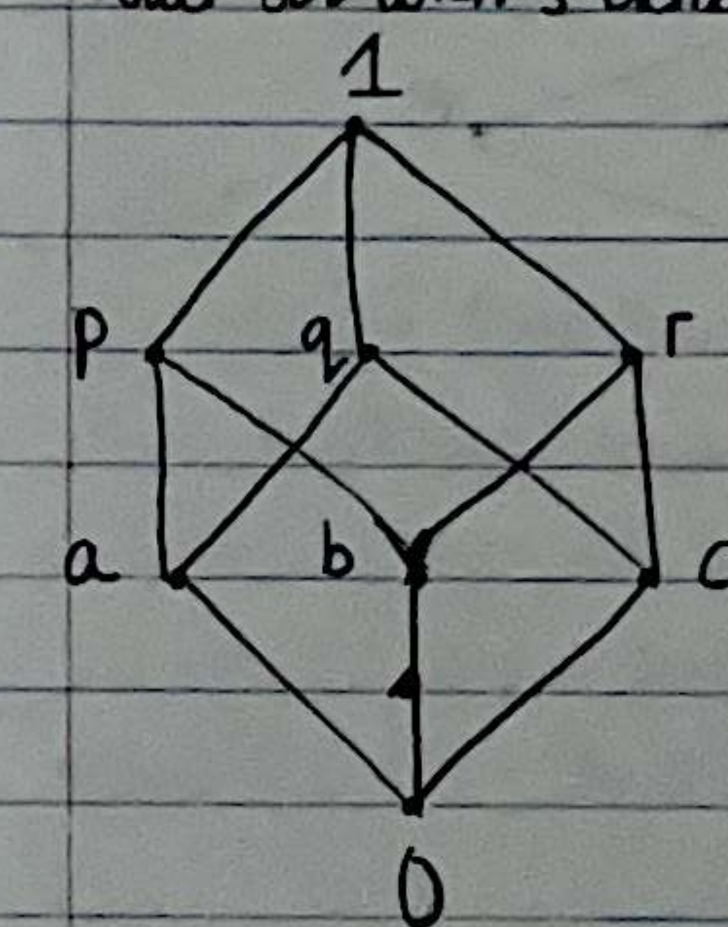
NOT 0 & 1

$c' = \{a\}$ → we have seen b is not complement of c (and vice versa)

$\Rightarrow$ only a

4) Distributive lattice

Power set with 3 elements under inclusion.



Find p' (TEST)

① Start with "close" elements (a & b)

$p \wedge a = a$ $p \vee a = p$ $\Rightarrow a$ is not complement

$p \wedge b = b$ $p \vee b = p$ $\Rightarrow b$ is not complement

② $p \wedge 0 = 0$ $p \vee 0 = 0$ $\Rightarrow$ Condition fails 0 is not complement.

③ $p \wedge c = 0$ $p \vee c = 1$

For p & c the meet is 0

For p & c the meet is 1

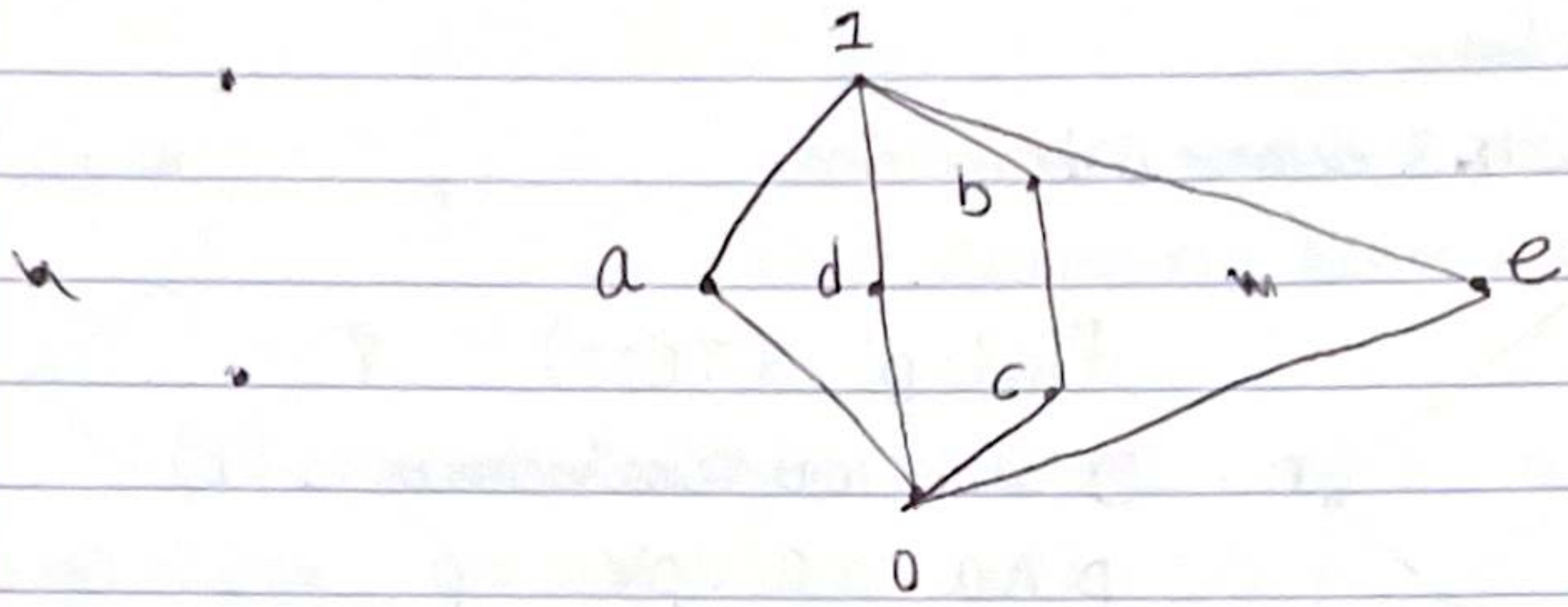
$\Rightarrow p' = c$

$\Rightarrow$ For this lattice the diagonally opposed element are complements.

$\Rightarrow a' = r$

$\Rightarrow q' = p$

5) The complements of the element a in the lattice shown in the figure is/are ___.



① $a \wedge d = 0 \quad a \vee d = 1 \quad \Rightarrow a' = d$

$a \wedge b = 0 \quad a \vee b = 1 \quad \Rightarrow a' = b$

$a \wedge c = 0 \quad a \vee b = 1 \quad \Rightarrow a' = c$

$a \wedge e = 0 \quad a \vee e = 1 \quad \Rightarrow a' = e$

$\Rightarrow a' = \{b, c, d, e\}$

This intro to lattice will lead to the Boolean Algebra which can be described as a bounded, complemented and distributive lattice.
Meet / Join ⟶ AND / OR.

Defn

Group Theory Cont.

-Af

Definition of

① Mapping

Defn 1

A mapping from S to T
of S with only one e

Defn 2

A mapping f from S t
$G \subseteq S \times T$ Such tha
$\forall x \in S : \forall y_1, y_2 \in T$
in words: If x f ser
(and $\forall x \in S : \exists y \in$

Defn 3

A mapping f from S t
$R \subseteq S \times T$ such th
$\forall (x_1, y_1)(x_2, y_2)$
and $\forall x \in S : \exists y \in$
↳ NOTE: Suppose
$f : S \to T : f(x$

NOTE:
(As can be seen, bi

Defn 4

A mapping from S
1) Many-to-One
2) Left-total.