Proof of Irrationality of $\sqrt{2}$ (by infinite descent)

Aiming for contradiction,

assume $\sqrt{2}$ is rational

$\Rightarrow \sqrt{2} = \frac{a}{b}$ where the ratio $a/_b$ is irreducible — ①

$\left(\frac{a}{b}\right)^2 = 2$

$\frac{a^2}{b^2} = 2$

$a^2 = 2b^2$ — ⓐ

$\Rightarrow a^2$ is even.

Lemma ①

Suppose $n = 2m$ is an even integer

$\Rightarrow n^2 = 4m^2$

$= 2(2m^2)$

$\Rightarrow n^2$ is even if n is even.

From Lemma 1 a is even and can be expressed as

$a = 2c$

Substituting in ⓐ

$(2c)^2 = 2b^2$

$\Rightarrow 2b^2 = 4c^2$

$\Rightarrow b^2 = 2c^2$

$\Rightarrow b^2$ is even

From lemma ①

$\Rightarrow$ b is even and can be expressed as

$b = 2d$

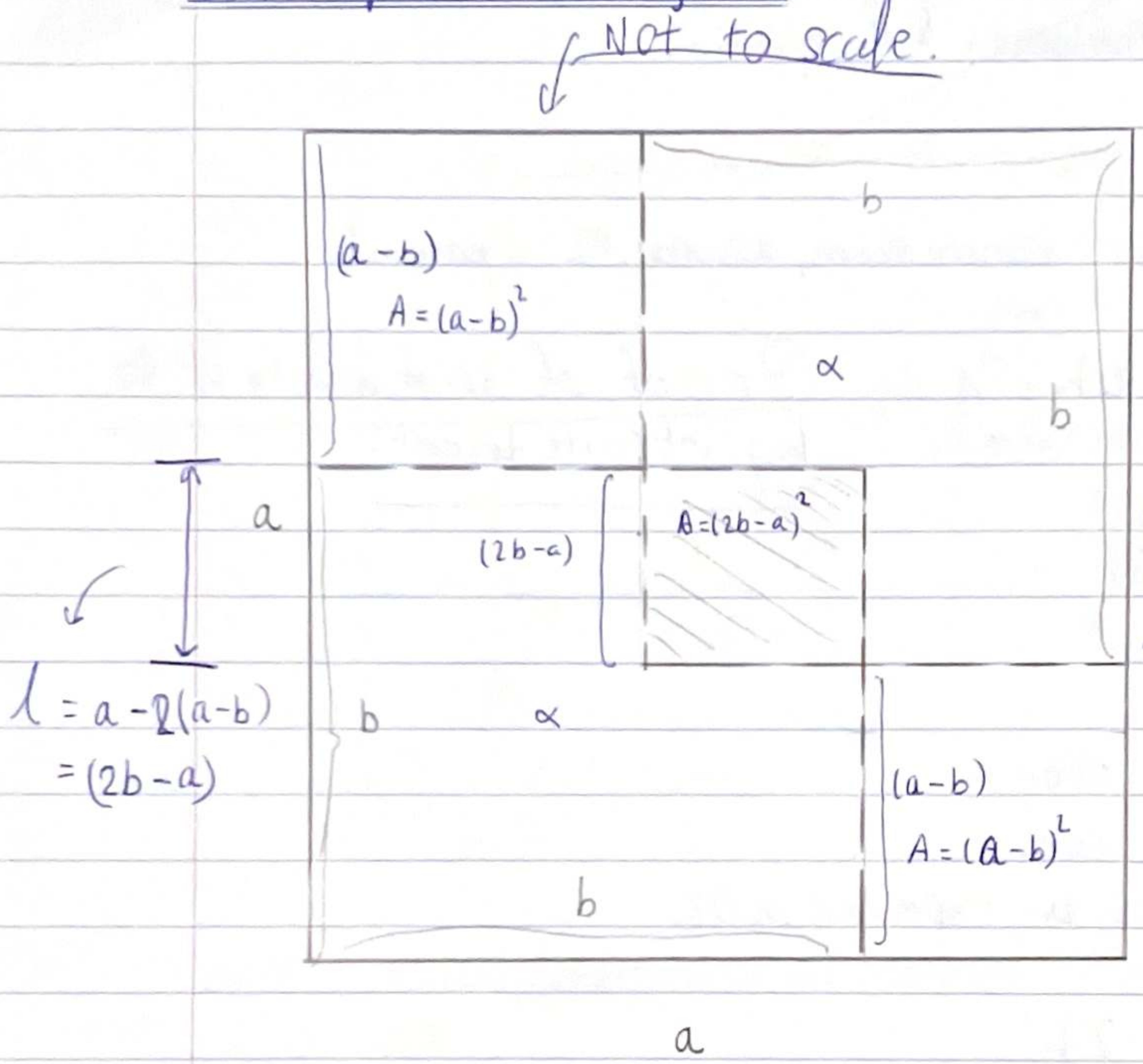
$\Rightarrow \frac{a}{b} = \frac{2c}{2d} = \frac{c}{d}$

As shown $a/_b$ is reducible which contradicts ①

Hence $\sqrt{2}$ is irrational.

∎

Geometric proof of irrationality of $\sqrt{2}$



(3)

Given 2 Squares with integer side a and b with the larger one having twice the area of the smaller one.

β = area ~~large~~ small square

$\alpha = \beta/2$ (large area)

$2\,\beta/2 = \alpha$

The overlapping region in the middle must have ~~twice~~ area equal to the sum of the uncovered regions.

~~$2\alpha = \beta$~~

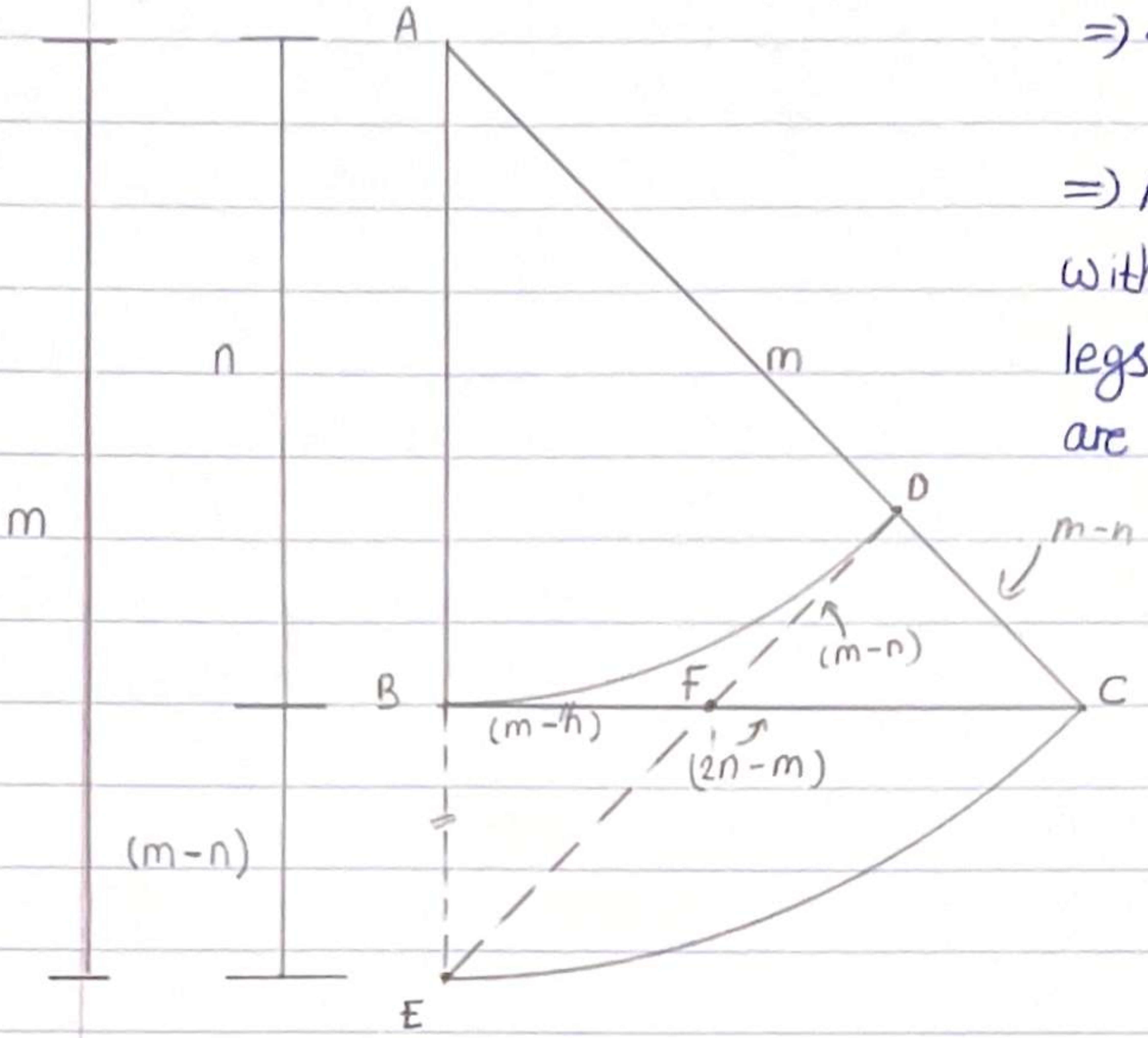
Residue of intersection area can be "moved" to cover unshaded parts.

$\Rightarrow (2b-a)^2 \stackrel{!}{=} 2(a-b)^2$

However the small squares ~~form~~ formed have integer sides smaller than the original squares.

⋮

Repeating for smaller squares until 1 is reached.



Since $DFC = 45°$

$\Rightarrow \triangle DFC$ is also isoceles.

$\Rightarrow$ A smaller isoceles $\triangle$ with hypotenuse $2n-m$ and legs $m-n$. These values are even smaller integers than m and n while being in the same previous ratio.

This is a contradiction.

$\triangle ABC$ is a right isoceles triangle with hypotenuse m and legs n

$\Rightarrow m = n\sqrt{2}$

$\Rightarrow \frac{m}{n} = \sqrt{2}$

Suppose m and n are integers and $m:n$ is in the simplest ratio. (lowest terms).

BD and CE are arcs with centre A.

$\Rightarrow AB = AD$ and $AE = AC$

$\angle BAC$ and $\angle DAE$ coincide

$\therefore \triangle ABC$ and $\triangle ADE$ are congruent by SAS

Because $\angle EBF$ is a right-angle and $\angle BEF$ is half a right-angle

$\Rightarrow \triangle BEF$ is also ~~a right~~ an isoceles triangle. $\rightarrow$ (Why??)

By Symmetry of isoceles $\triangle$ in a kite

$\Rightarrow DF = m - n$

$\triangle ABC$ is congruent with $\triangle ADE$.