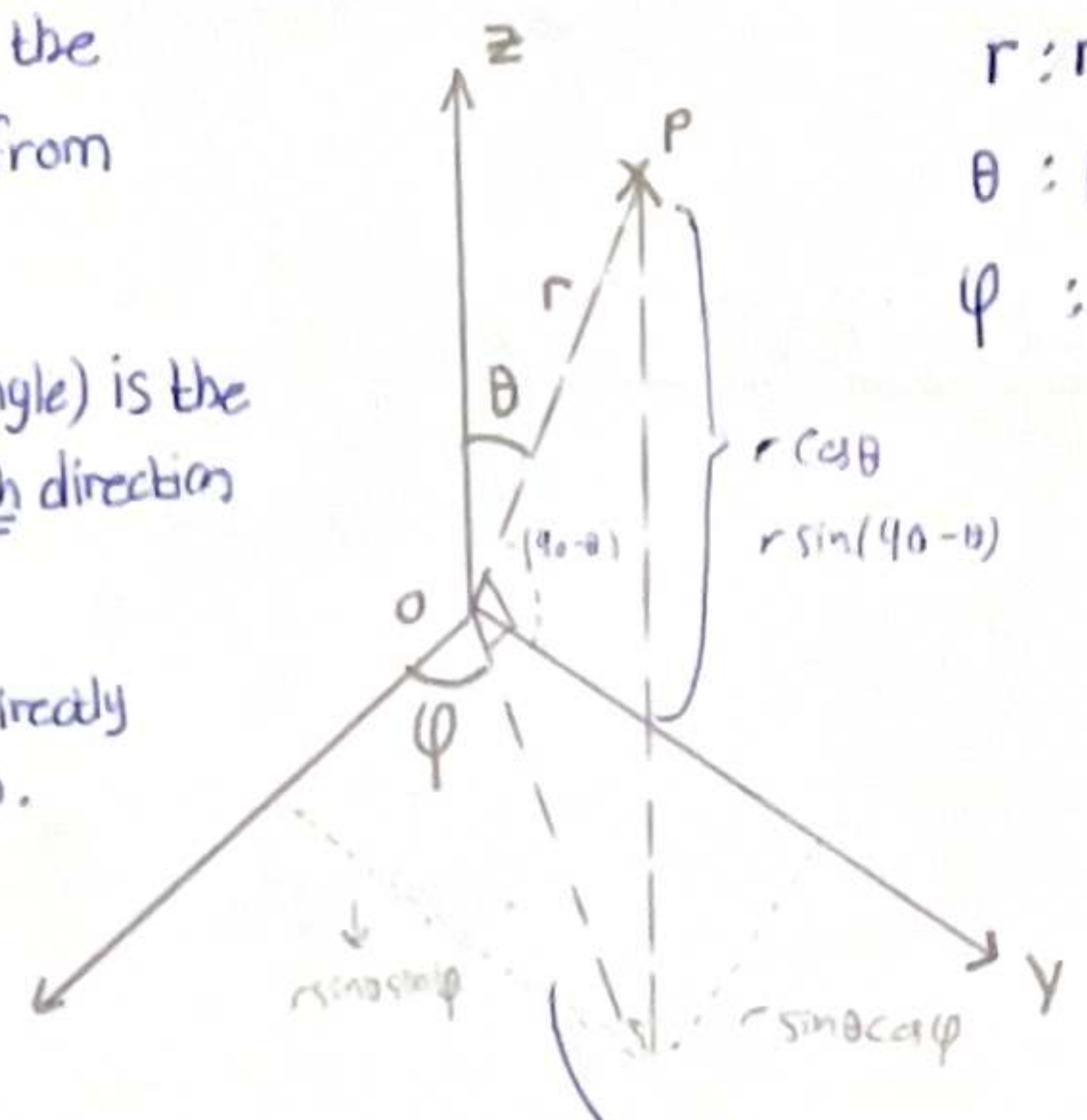
Spherical-Coordinate System

r : radius or radial distance is the ~~eucl~~ euclidean distance from the origin O to P.

θ : The inclination (or polar angle) is the angle between the zenith direction and the line segment OP.

Point directly above O.



orthogonal projection of OP.

r : radial distance, (or ρ)

θ : Polar angle.

φ : Azimuthal angle.

horizontal angle or direction of a compass bearing.

φ : Azimuth or azimuthal angle is the signed angle measured from the azimuth reference direction to the orthogonal projection of line segment OP on the reference plane. The sign of the azimuth is determined by choosing what is a positive sense of turning about the zenith.

The elevation angle is 90° or $\frac{\pi}{2}$ rad minus the inclination angle.

Plotting

To plot a dot from its spherical coordinates, (r, θ, φ) where θ is inclination, move r units from the origin into the zenith direction rotate by θ about the origin towards the azimuth reference and rotate by φ about the zenith in the proper direction.

Coordinate-System Conversion

Cartesian Coordinates to Polar Coordinates

$(x, y, z) \rightarrow (r, \theta, \varphi)$

- $r = \sqrt{x^2 + y^2 + z^2}$
- $\theta = \underbrace{\arccos}_{\cos^{-1}} \dfrac{z}{\sqrt{x^2+y^2+z^2}} = \cos^{-1}\left(\dfrac{z}{r}\right)$
- $\varphi = \underbrace{\arctan}_{\tan^{-1}} \left(\dfrac{y}{x}\right)$

Polar Coordinate to Cartesian Coordinate.

$r \in [0, \infty) \quad \theta \in [0, \pi], \; \varphi \in [0, 2\pi)$

- $x = r \sin\theta \cos\varphi$
- $y = r \sin\theta \sin\varphi$
- $z = r \cos\theta$

Equation of sphere in Cartesian Coordinate.

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \rightarrow \text{or general } x^2+y^2+z^2=r^2$$

Cartesian equation.

defined as $F(x,y,z)$

Where a, b and C are the coordinates for the Centre of the sphere and r is the radius of the sphere.

Volume of a Sphere

Polar equation

Note $r^2 \sin\theta$

defined as $G(r,\theta,\varphi)$

$V = x \times y \times z$

$dV = dx\,dy\,dz$

from spherical coordinates,

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

$V = r\sin\theta\cos\varphi \times r\sin\theta\sin\varphi \times r\cos\theta$

Jacobian required to proceed.

Evaluation of third-order determinant

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow \det A = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$$

Jacobian Matrix and determinant

Change of Variable in Multiple Integrals

x & y are functions

The jacobian of the transformation T given by $x = g(u,v)$ and $y = h(u,v)$ is :

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \overset{\text{det.}}{=} \left(\frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v}\right) - \left(\frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}\right)$$

$$x = u^2 - v^2$$
$$y = u^2 + v^2$$

v is constant = 0

$$\frac{\partial x}{\partial u} = 2u \cancel{v^2} \qquad \frac{\partial x}{\partial v} = -2v$$

$$\frac{\partial y}{\partial u} = 2u \qquad \frac{\partial y}{\partial v} = 2v$$

$$\Rightarrow \begin{vmatrix} 2u & -2v \\ 2u & 2v \end{vmatrix} \Rightarrow \text{det.} = (2u \times 2v) - (-2v \times 2u)$$
$$= 4uv + 4uv$$
$$= 8uv$$

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eg. double integral

$$\iint_R f(x,y)\,dA = \iint F(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

Evaluate : $\iint_R (x - 3y)\,dA$ where R is the triangular region with vertices $(0,0)$, $(2,1)$ and $(1,2)$ using the transformation $x = 2u + v$, $y = u + 2v$

$$\frac{\partial x}{\partial u} = 2 \qquad \frac{\partial x}{\partial v} = 1$$

$$\frac{\partial y}{\partial u} = 1 \qquad \frac{\partial y}{\partial v} = 2$$

absolute value

$$\Rightarrow \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \quad \text{det.} = |(4) - (1)| = |3| = 3$$

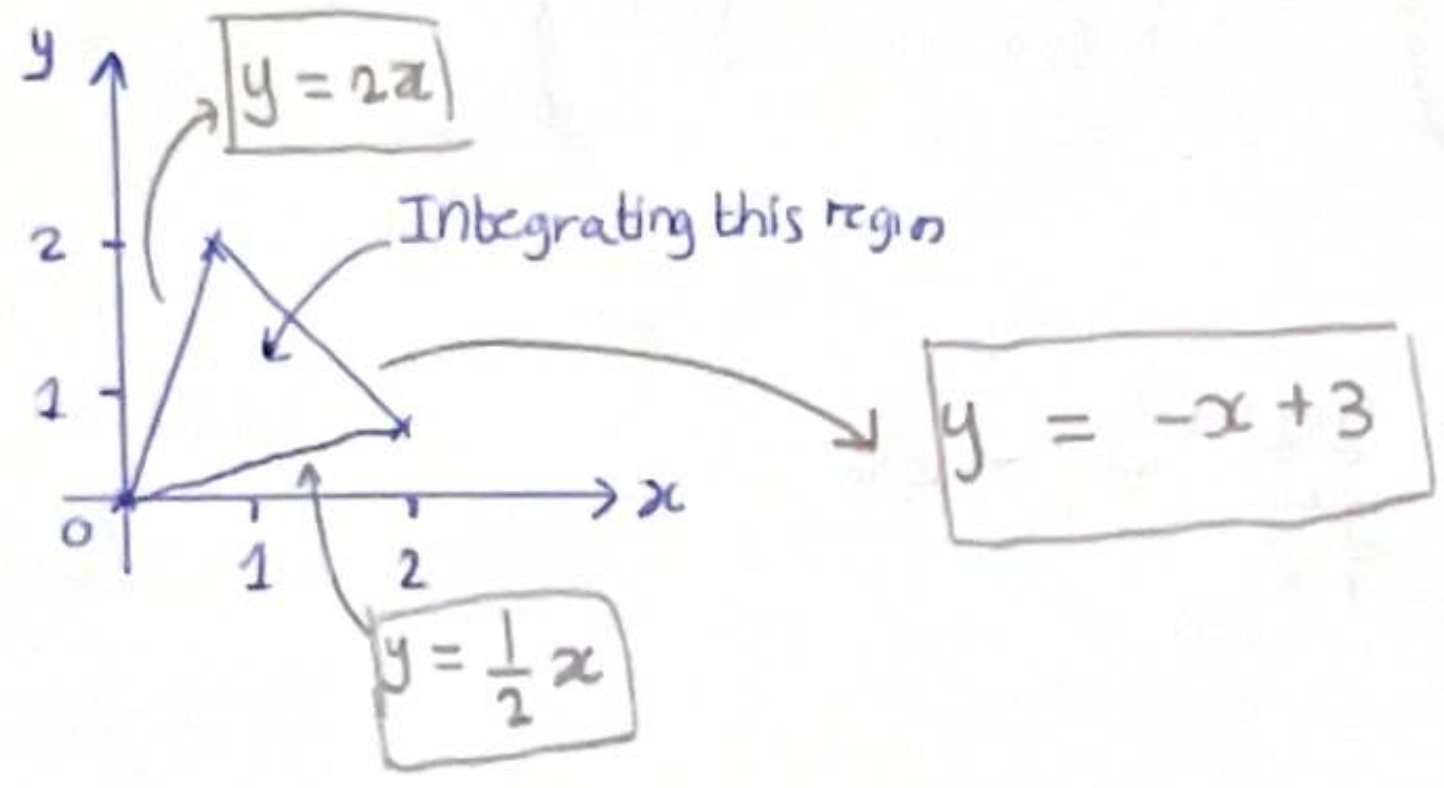
+ fourier Series
+ eigenvalue ✓
eigenvector ✓
Hyperbolic. ✓
linear transformation
Hamiltonian
Hilbert Space
Particular integral
~~Integral~~ Integrating factor
Matrix
Set interval notation. ✓
Separation of variables

$\Rightarrow \iint (x - 3y)$ from $x = 2u + v \quad y = u + 2v$

$$\Rightarrow [2u + v - 3(u + 2v)] = [2u + v - 3u - 6v] = [-u - 5v]$$

$$\iint_{?} -u - 5v\ |3|\ du\, dv$$

Solve for R.



$x = 2u + v \qquad y = u + 2v$

$$\left.\begin{aligned} y &= 2x \\ y &= \tfrac{1}{2}x \\ y &= -x + 3 \end{aligned}\right\}$$ Apply above transformation

$y = 2x$

$(u + 2v) = 2(2u + v)$

$u + \cancel{2v} = 4u + \cancel{2v}$

$3u = 0$

$\Rightarrow u = 0 \leftarrow$ Transformed

$y = \frac{1}{2}x$

$(u + 2v) = \frac{1}{2}(2u + v)$

$\cancel{2u} + 4v = \cancel{2u} + v$

$3v = 0$

$v = 0$

$y = -x + 3$

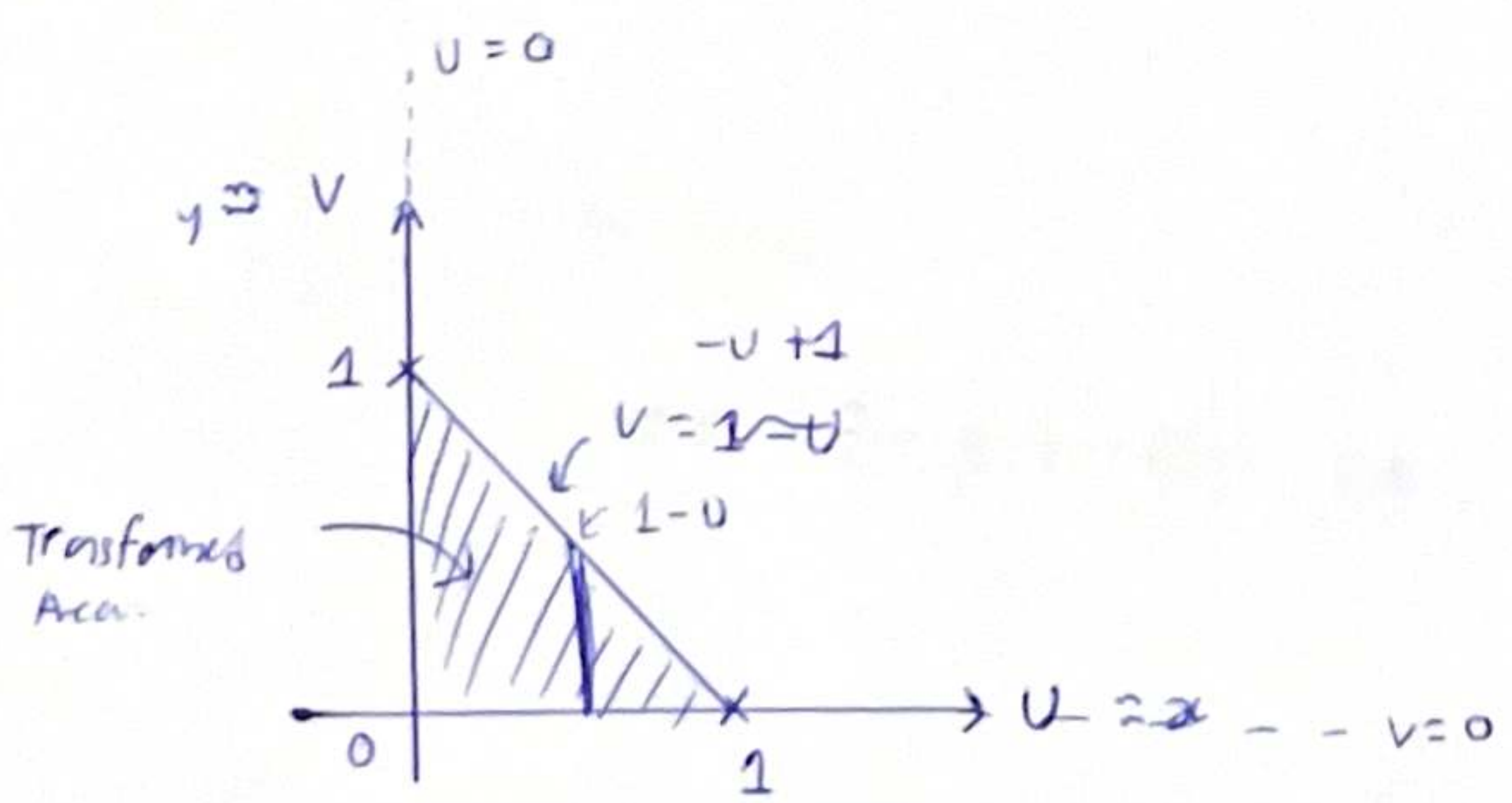
$u + 2v = -(2u + v) + 3$

$= -2u - v + 3$

$3u + 3v = 3$

$u + v = 1$

$v = -u + 1$



$$\Rightarrow 3\int_0^1\int_0^{1-u}(-u-5v)\,dv\,du$$

Solve for limit of integration from graph.

Range of V : 0 to $1-U$

U : 0 to 1

Integrating w.r.t v first.

$$= -3\int_0^1\int_0^{1-u}(u+5v)\,dv\,du$$

$$= -3\int_0^1\left[uv + \frac{5v^2}{2}\Big|_{v=0}^{v=1-u}\right]du$$

$$= -3\int_0^1\left(\left[u(1-u)+\frac{5}{2}(1-u)^2\right] - [0]\right)du$$

$$= -3\int_0^1\left[u-u^2+\frac{5}{2}(1-2u+u^2)\right]du$$

$$= -3\int_0^1\left[u-u^2+\frac{5}{2}-5u+\frac{5}{2}u^2\right]du$$

$$= -3\int_0^1\left(\frac{5}{2}-4u+\frac{3}{2}u^2\right)du$$

$$= -3\left[\frac{5}{2}u-2u^2+\frac{u^3}{2}\Big|_0^1\right]$$

$$= -3\left[\left(\frac{5}{2}-2+\frac{1}{2}\right)-0\right]$$

$$= -3(\overset{+1}{3-2})$$

$= -3$ ← Value of original Integral.

Volume of a sphere

Transform 3 dimensional Cartesian Coordinate System (x,y,z) to Spherical polar Coordinate System (r,θ,φ)

A change of Variable involves defing x, y and z as a function of r, θ and φ

Relationship

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

$x = f(r,\theta,\varphi) = r\sin\theta\cos\varphi$

$y = g(r,\theta,\varphi) = r\sin\theta\sin\varphi$

$z = h(r,\theta,\varphi) = r\cos\theta$

$$\underbrace{\begin{array}{l} r\in[0,\infty) \\ \theta\in[0,\pi] \\ \varphi\in[0,2\pi) \end{array}}_{\text{limits defined}}$$

Volume element in Cartesian Coordinate is : $dV = dx\,dy\,dz$

and is calculated using : Volume $= \iiint F(x,y,z)\,dx\,dy\,dz$

In polar coordinates : Volume $= \iiint G(r,\theta,\varphi)\,|J|\,dr\,d\theta\,d\varphi$ *

Where J is the Jacobian which is the ~~mat~~ determinant of the matrix that transforms one coordinate into another.

$$dx = \frac{\partial f(r,\theta,\varphi)}{\partial r}dr + \frac{\partial f(r,\theta,\varphi)}{\partial \theta}d\theta + \frac{\partial f(r,\theta,\varphi)}{\partial \varphi}d\varphi$$

$$dy = \frac{\partial g(r,\theta,\varphi)}{\partial r}dr + \frac{\partial g(r,\theta,\varphi)}{\partial \theta}d\theta + \frac{\partial g(r,\theta,\varphi)}{\partial \varphi}d\varphi$$

$$dz = \frac{\partial h(r,\theta,\varphi)}{\partial r}dr + \frac{\partial h(r,\theta,\varphi)}{\partial \theta}d\theta + \frac{\partial h(r,\theta,\varphi)}{\partial \varphi}d\varphi$$

In matrix

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} dr \\ d\theta \\ d\varphi \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \varphi} \\ \frac{\partial g}{\partial r} & \frac{\partial g}{\partial \theta} & \frac{\partial g}{\partial \varphi} \\ \frac{\partial h}{\partial r} & \frac{\partial h}{\partial \theta} & \frac{\partial h}{\partial \varphi} \end{pmatrix} \leftarrow \text{det.} = \frac{\partial f}{\partial r}\begin{vmatrix} \frac{\partial g}{\partial \theta} & \frac{\partial g}{\partial \varphi} \\ \frac{\partial h}{\partial \theta} & \frac{\partial h}{\partial \varphi} \end{vmatrix} - \frac{\partial f}{\partial \theta}\begin{vmatrix} \frac{\partial g}{\partial r} & \frac{\partial g}{\partial \varphi} \\ \frac{\partial h}{\partial r} & \frac{\partial h}{\partial \varphi} \end{vmatrix} + \frac{\partial f}{\partial \varphi}\begin{vmatrix} \frac{\partial g}{\partial r} & \frac{\partial g}{\partial \theta} \\ \frac{\partial h}{\partial r} & \frac{\partial h}{\partial \theta} \end{vmatrix}$$

$f(r,\theta,\varphi) = r\sin\theta\cos\varphi$

$g(r,\theta,\varphi) = r\sin\theta\sin\varphi$

$h(r,\theta,\varphi) = r\cos\theta$

$\left(\frac{\partial f}{\partial r}\right)_{\theta,\varphi} = \sin\theta\cos\varphi \qquad \left(\frac{\partial f}{\partial \theta}\right)_{r,\varphi} = r\cos\theta\cos\varphi \qquad \left(\frac{\partial f}{\partial \varphi}\right)_{r,\theta} = -r\sin\theta\sin\varphi$

$\left(\frac{\partial g}{\partial r}\right)_{\theta,\varphi} = \sin\theta\sin\varphi \qquad \left(\frac{\partial g}{\partial \theta}\right)_{r,\varphi} = r\cos\theta\sin\varphi \qquad \left(\frac{\partial g}{\partial \varphi}\right)_{r,\theta} = r\sin\theta\cos\varphi$

$\left(\frac{\partial h}{\partial r}\right)_{\theta,\varphi} = \cos\theta \qquad \left(\frac{\partial h}{\partial \theta}\right)_{r,\varphi} = -r\sin\theta \qquad \left(\frac{\partial h}{\partial \varphi}\right)_{r,\theta} = 0$

$\Rightarrow |J| = \sin\theta\cos\varphi\left[r^2\sin^2\theta\cos\varphi\right] - r\cos\theta\cos\varphi\left[-r\sin\theta\cos\theta\cos\varphi\right] - r\sin\theta\sin\varphi\left[-r\sin^2\theta\sin\varphi - r\cos^2\theta\sin\varphi\right]$

ie det. J

Using $\cos^2\theta + \sin^2\theta = 1$

$|J| = r^2\sin^2\theta\cos^2\varphi\sin\theta + r^2\cos^2\theta\cos^2\varphi\sin\theta + r^2\sin\theta\sin^2\varphi$

$= r^2\sin\theta\left[\sin^2\theta\cos^2\varphi + \cos^2\theta\cos^2\varphi + \sin^2\varphi\right]$

$= r^2\sin\theta \leftarrow$

In spherical coordinates,

$dV = r^2\sin\theta\, dr\, d\theta\, d\varphi$

or

$dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\varphi$

new function after transformation

$\Rightarrow \text{Volume} = \iiint G(r,\theta,\varphi)\, r^2\, dr\, \sin\theta\, d\theta\, d\varphi$

or $\iiint G(r,\theta,\varphi)\, r^2\sin\theta\, dr\, d\theta\, d\varphi \leftarrow$

with limits : $r \in [0,\infty)$

$\theta \in [0,\pi]$

$\varphi \in [0,2\pi)$