Cross Product

$\vec{A} \times \vec{B} = \vec{C}$ where $C \perp (A \wedge B)$

$$\begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \end{array}$$

$$\begin{vmatrix} e_x & e_y & e_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$= e_x \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - e_y \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + e_z \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$(A_yB_z - A_zB_y)e_x - (A_xB_z - A_zB_x)e_y + (A_xB_y - A_yB_x)e_z$$

Levi-civita symbol.

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{- Cyclic permutation , 1,2,3 / 2,3,1, / 3,1,2} \\ -1 & \text{- Anticyclic permutations - 321, 213, 132} \\ 0 & \text{- else (repeat) 1,1,3 , 3,1,3} \end{cases}$$

$i = 1 \rightarrow$ x component

$i = 2 \rightarrow$ y component

$i = 3 \rightarrow$ z component

Consider : $(A_yB_z - A_zB_y)e_x$

$$(\vec{A} \times \vec{B})_i = \sum_j \sum_k \varepsilon_{ijk} A_j B_k$$

$i = 1$

$$(\vec{A} \times \vec{B})_1 = \sum_j \sum_k \varepsilon_{1jk} A_j B_k$$

$$= \varepsilon_{111} \underbrace{A_1B_1}_{0} + \varepsilon_{112} \underbrace{A_1B_2}_{0} + \varepsilon_{113} A_1B_3$$

$$+ \varepsilon_{121} A_2B_1 + \varepsilon_{122} A_2B_2 + \varepsilon_{131} A_3B_1$$

$$+ \varepsilon_{123} A_2B_3 + \varepsilon_{132} A_3B_2 + \varepsilon_{133} A_3B_3$$

$$\cancel{= \varepsilon_{111} A_1B_1 + \varepsilon_{112} A_2B_2 + \varepsilon_{113} A_3B_3}$$

$$= \varepsilon_{111} \underbrace{A_1B_1}_{0} + \varepsilon_{112} \underbrace{A_1B_2}_{0} + \varepsilon_{113} \underbrace{A_1B_3}_{0}$$

$$+ \varepsilon_{121} \underbrace{A_2B_1}_{0} + \varepsilon_{122} \underbrace{A_2B_2}_{0} + \varepsilon_{131} \underbrace{A_3B_1}_{0}$$

$$+ \underline{\varepsilon_{123} A_2B_3} + \underline{\varepsilon_{132} A_3B_2} + \varepsilon_{133} \underbrace{A_3B_3}_{0}$$

$$\underbrace{\varepsilon_{123}}_{\text{cyclic}} A_2 B_3 + \underbrace{\varepsilon_{132}}_{\text{A-cyclic}} A_3 B_2$$

$$(\vec{A} \times \vec{B})_1 = A_2 B_3 - A_3 B_2$$

$$= A_y B_z - A_z B_y$$

$$\cancel{(\vec{A} \times \vec{B})_2 = \varepsilon}$$

$$-(A_x B_z - A_z B_x) = (A_x B_z + A_z B_x)$$

$$(\vec{A} \times \vec{B})_2 = \sum_j \sum_k \varepsilon_{2jk}$$

$$\hookrightarrow \varepsilon_{2\underbrace{jk}_{j\,k\text{ cannot be }2}} A_j B_k$$

one of these has to be 1 or 3

$$\varepsilon_{231} A_3 B_1 + \varepsilon_{213} A_1 B_3$$

$$= A_3 B_1 - A_1 B_3$$

$$A_z B_x - A_x B_z$$

$$(\vec{A} \times \vec{B})_3 = \varepsilon_{3\underbrace{jk}_{1\text{ or }2}}$$

$$\varepsilon_{312} A_1 B_2 + \varepsilon_{321} A_2 B_1$$

$$A_1 B_2 - A_2 B_1$$

$$(A_x B_y - A_y B_x)$$

1 2 3 1 2 3 1 2 3

123, 231, 312

$$(\vec{A}\times\vec{B}) = \begin{array}{c|ccc|} & \hat{i} & \hat{j} & \hat{k} \\ A & 1 & 2 & 3 \\ B & 3 & 2 & 1 \end{array}$$

123 123 123

123 , 231 , ~~321~~ 312

$(\vec{A}\times\vec{B})_1 = \varepsilon_{123} + \varepsilon_{132}$

$= 1(2\times1) - 1(3\times2)$

$= 2 - 6$

$= -4$

$\hat{j}\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = \hat{j}\,|1 - 9| = -8\hat{j}$

$\hat{i}\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = 2 - 6 = -4$

$(\vec{A}\times\vec{B})_2 = \varepsilon_{213} + \varepsilon_{231}$

$= -1(1\times1) + 1(3\times3)$

$= -1 + 9$

$= -8$

$\hat{j}\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 1 - 9$

$= -8$

$(\vec{A}\times\vec{B})_3 = \varepsilon_{312} + \varepsilon_{321}$

$= +(1\times2) \mp 1(2\times3)$

$= +2 \; \bar{+}6$

$= -4$

$= k\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} = 2 - 6$

$= -4$

$$A = \begin{array}{c|ccc|} & 1 & 2 & 3 \\ A & 3 & 2 & 1 \\ B & 1 & 3 & 2 \end{array}$$

$(\vec{A} \times \vec{B}) =$

123 123 123

123 231 312

$$(A \times B)_1 = \cancel{\varepsilon_{123} + \varepsilon_{132}}$$

$$\cancel{= 1(2\times 3)\times(+1) + 1(1\times 3)(-1)}$$

$$\cancel{= 6 - 3 = 3}$$

$$1\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = \underline{1}$$

$$(A \times B)_1 = 1\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 1 \leftarrow$$

Einstein Convention.

$$\det(A) = \cancel{\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}} \varepsilon_{ijk}\, a_{1i}\, a_{2j}\, a_{3k}$$

$$= \varepsilon_{123} + \varepsilon_{132}$$

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix}$$

$$\det(A) = \cancel{\varepsilon_{132} + \varepsilon_{123} + \varepsilon}$$

$$\cancel{\varepsilon_{123} + \varepsilon_{132} + \varepsilon_{213} + \varepsilon_{231} + \varepsilon_{312} + \varepsilon_{321}}$$

$\cancel{1}(A)$

$$(A \times B) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$$

123
1231
312

$(A \times B)_1 = \varepsilon_{123} + \varepsilon_{132}$

$= (1)(2 \times 1) + (-1)(3 \times 2)$

$= 2 - 6$

$= -4$

$$\begin{pmatrix} -4 \\ 8 \\ -4 \end{pmatrix}$$

$(A \times B)_2 = \varepsilon_{213} + \varepsilon_{231}$

$= (-1)(1 \times 1) + (+1)(3 \times 3)$

$= -1 + 9$

$= 8$

$(A \times B)_3 = \varepsilon_{312} + \varepsilon_{321}$

$= (+1)(1 \times 2) + (-1)(2 \times 3)$

$= 2 - 6$

$= -4$

$$\therefore \begin{pmatrix} -4 \\ 8 \\ -4 \end{pmatrix}$$

Kronecker Delta

$$\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

$i = 1$
$j = 2$
$k = 3$

eg. $\delta_{12} = 0$, $\delta_{33} = 1$

eg $a.b$

$a = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $b = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$a.b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = (1 \times 3) + (2 \times 2) + (3 \times 1) = 10$

$$a.b = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \delta_{ij} b_j$$

$(a \cdot b)_1 = 1 \cdot (\delta_{11}) \cdot (3) = 3$

$(a \cdot b) =$

$$a.b = \begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}3\\2\\1\end{pmatrix} = (1\times3)+(2\times2)+(3\times1)$$
$$= 10$$

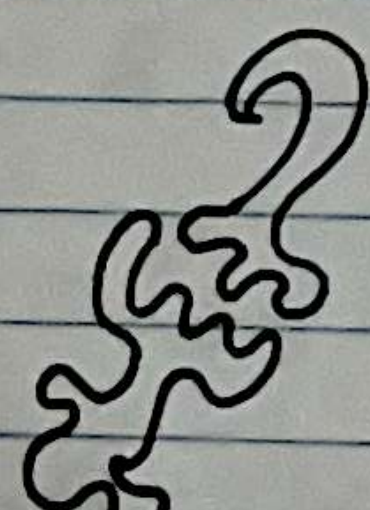
$$a.b = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} a_i$$

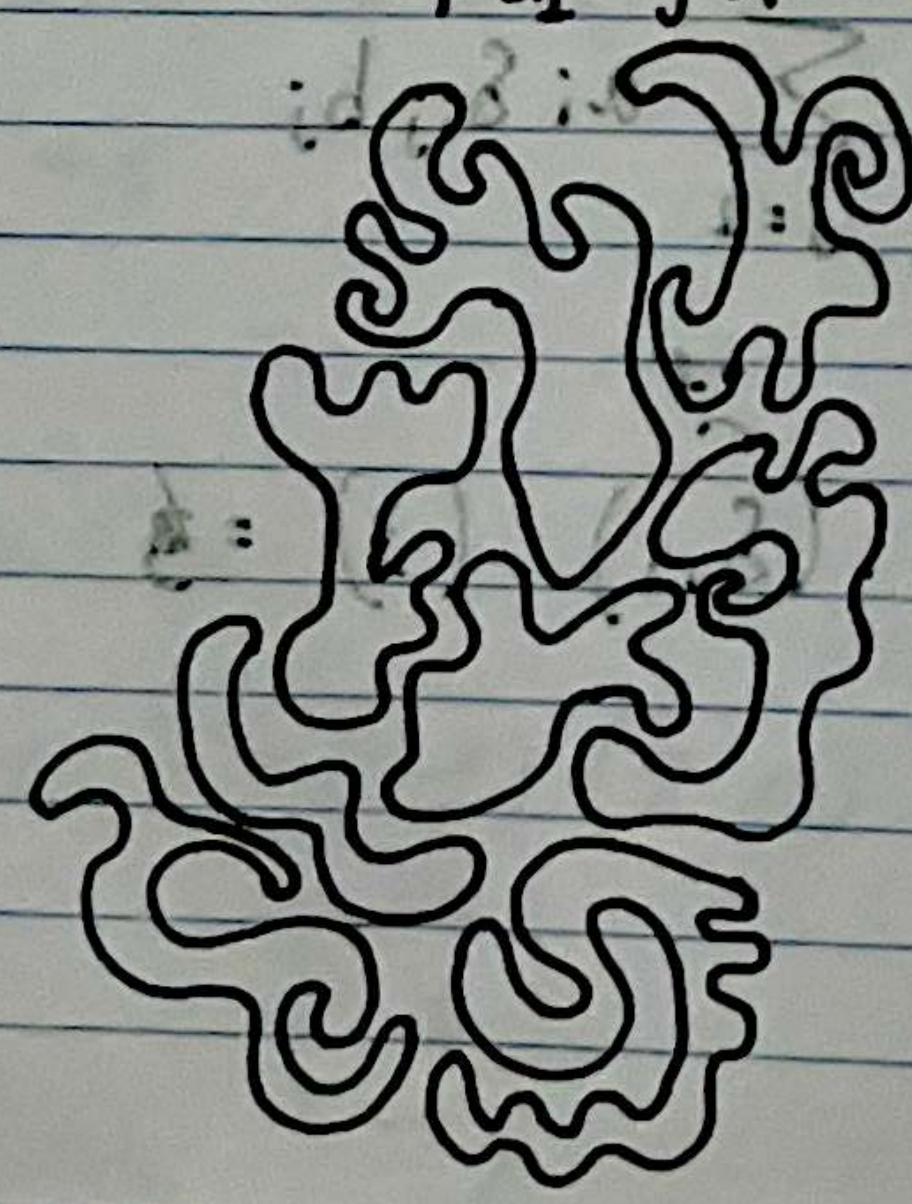
~~a.b~~

$$A = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \quad B = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

$$A.B = a_1b_1 + a_2b_2 + a_3b_3$$

$$= \sum_{i=1}^{3} a_i b_i = \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \delta_{ij}$$





$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 3 & 1\\ 2 & 1 & 3\end{vmatrix} \longrightarrow \hat{i}\begin{vmatrix}9-1\end{vmatrix} = 8$$

$$=8$$

$$-\hat{j}\begin{vmatrix}12-2\end{vmatrix} \quad 10$$

$$\hat{k}\begin{vmatrix}4-6\end{vmatrix} \quad -2$$

$$e_x = \varepsilon_{123} + \varepsilon_{132}$$

$$= (+1)(3\times3) + (-1)(1\times1)$$

$$= 9 - 1$$

$$= 8$$

$$\begin{pmatrix}8\\-10\\-2\end{pmatrix}$$

$$e_y = \varepsilon_{213} + \varepsilon_{231}$$

$$= (-1)(4\times3) + (+1)(1\times2)$$

$$= -12 + 2$$

$$= -10$$

$$e_z = \varepsilon_{312} + \varepsilon_{321}$$

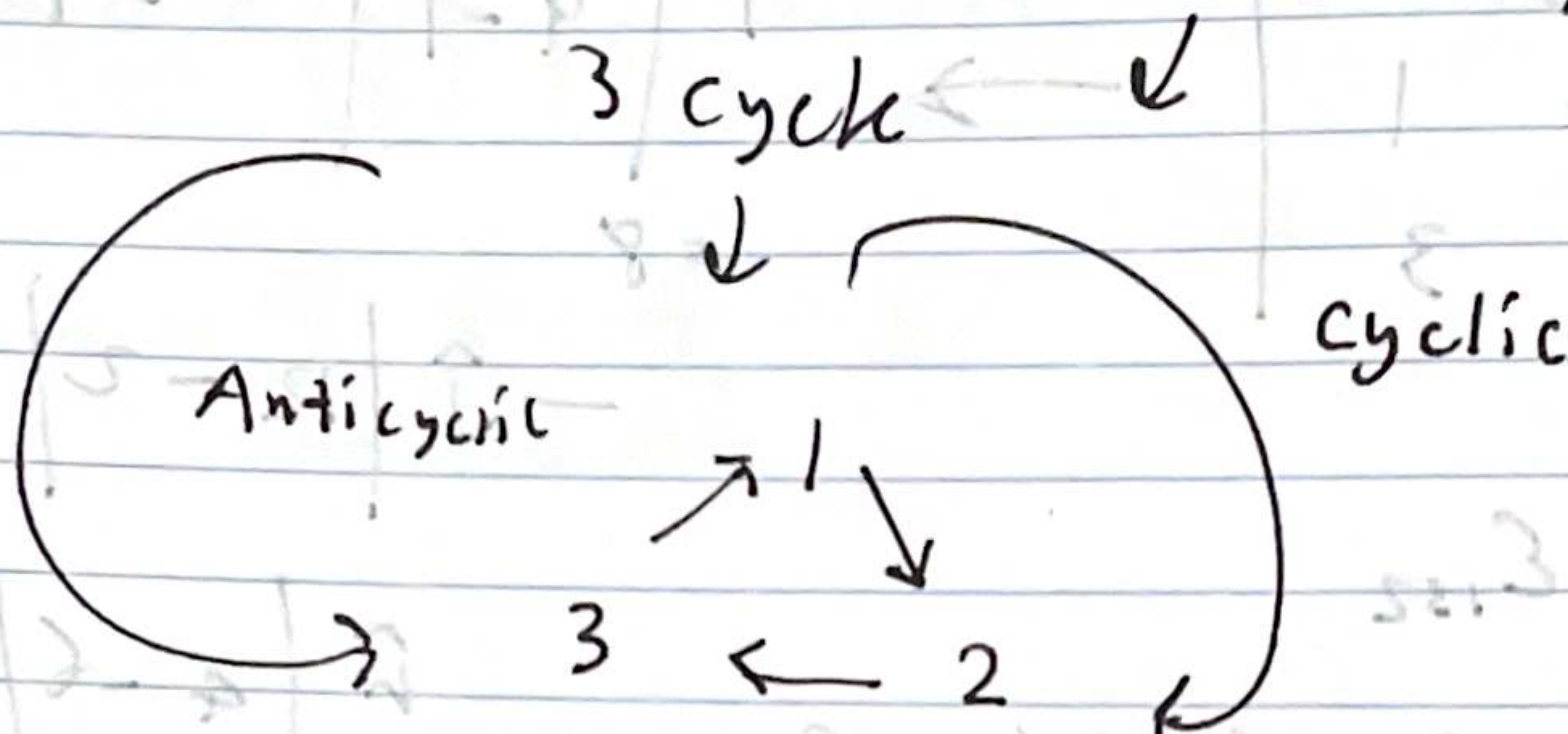
$$= (+1)(4\times1) + (-1)(3\times2)$$

$$= 4 - 6$$

$$= -2$$

$$\therefore \begin{pmatrix}8\\-10\\-2\end{pmatrix}$$

Cyclic permutation.



123 231 312 Cyclic

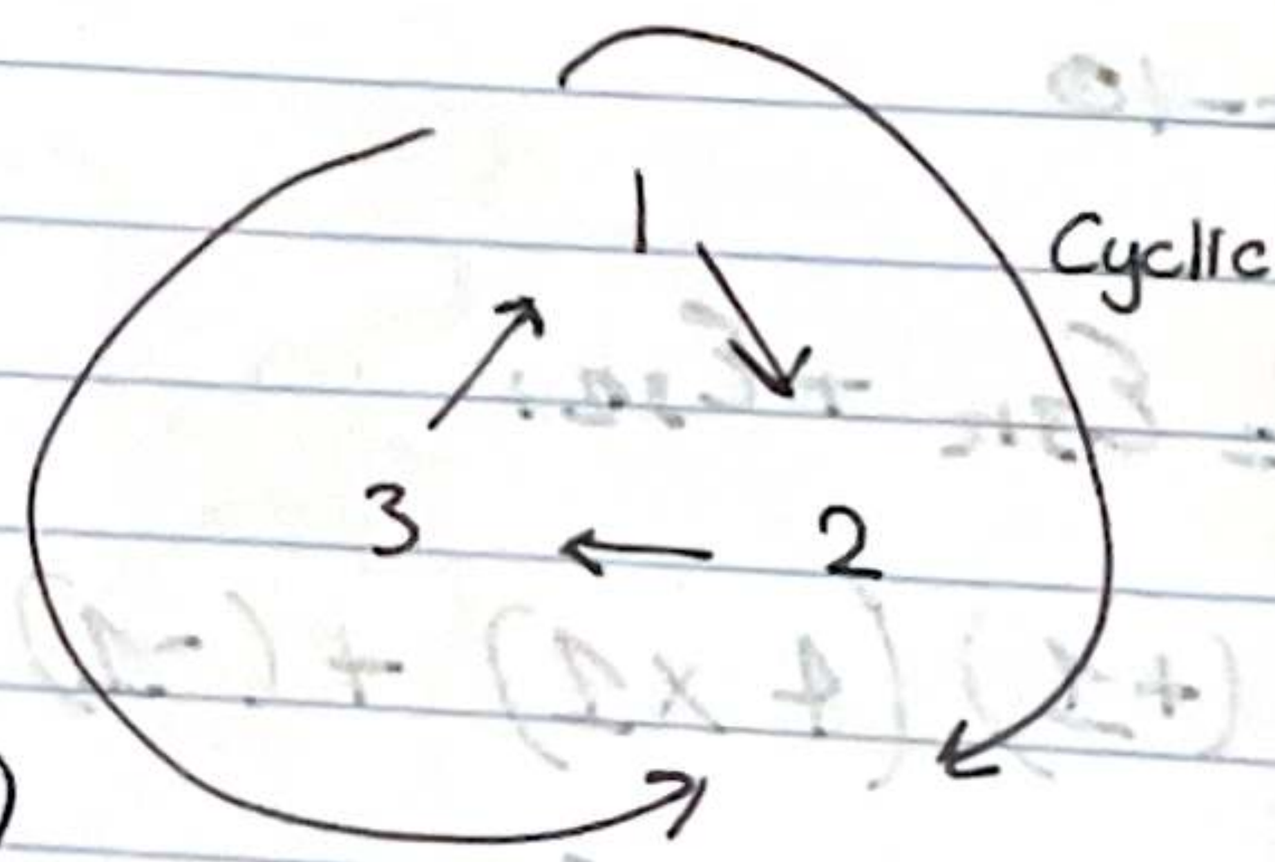
132 213 321 ~~Anticlic~~ Anticyclic.

$$\begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$$

-8

$\to -11$

Anticyclic.



$\hat{r}_i = \varepsilon_{123} + \varepsilon_{132}$

$= (+1)(2\times 1) + (-1)(3\times 2)$

$2 - 6$

$= -4$ ✓