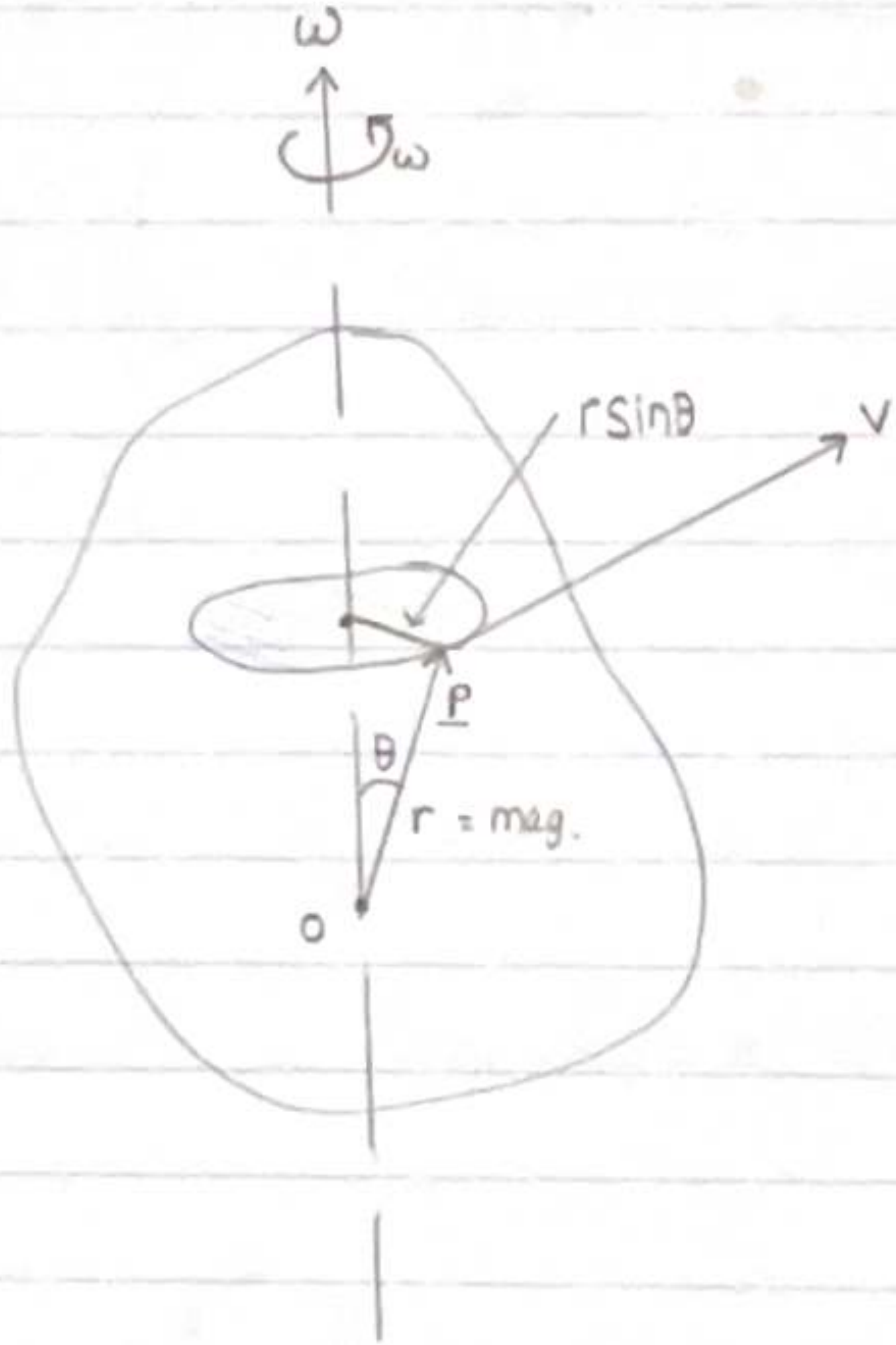
A rigid body rotates about an axis through point O with angular speed ω. Prove that linear velocity V of a point P of the body with position vector r is given by $V = \omega \times r$, where ω is the vector with magnitude ω whose direction is that in which a right-handed screw would advance under the given rotation.



P travels in a radius $r\sin\theta$, the magnitude of the linear velocity V is $|\omega \times r| = \omega r \sin\theta$

Also, V, ω and r are mutually perpendicular.

$\Rightarrow$ Both in direction and in magnitude, $V = \omega \times r$ (ω is the angular velocity)

Ordinary Integrals of Vectors.

Let $R(U) = R_1(U)\hat{i} + R_2(U)\hat{j} + R_3(U)\hat{k}$ be a vector depending on a Single Scalar Variable U, where $R_1(U), R_2(U), R_3(U)$ are Supposed continuous in a Specified interval.
Then,

$$\int R(U)\,dU = \hat{i}\int R_1(U)\,dU + \hat{j}\int R_2(U)\,dU + k\int R_3(U)\,dU$$ is called

an indefinite integral of $R(U)$. If there exists a Vector $S(U)$ Such that

$$R(U) = \frac{d}{dU}(S(U))$$, then,

$$\int R(U)\,dU = \int \frac{d}{dU}(S(U))\,dU = S(U) + C$$ Where C is an arbitrary Constant vector independant of U. The definite integral between limits $U=a$ and $U=b$ Can be written as

$$\int_a^b R(U)\,dU = \int_a^b \frac{d}{dU}(S(U))\,dU = S(U) + C\Big|_a^b$$

$$= S(b) - S(a)$$

This integral can also be defined as the limit of a sum.

Evaluate: $\int A \times \frac{d^2A}{dt^2}\, dt$

$\cancel{dt}\ \cancel{\frac{d}{dt^2}(A}$

$$\frac{d}{dt}\left(A \times \frac{dA}{dt}\right) = A \times \frac{d^2A}{dt^2} + \frac{dA}{dt} \times \frac{dA}{dt} = A \times \frac{d^2A}{dt^2}$$

Integrating, $\int A \times \frac{d^2A}{dt^2}\, dt = \int \frac{d}{dt}\left(A \times \frac{dA}{dt}\right) dt = A \times \frac{dA}{dt} + C$

Line Integral

$r(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}$ where $r(u)$ is the p.v (x,y,z), define a curve C joining point P_1 and P_2 where $u = u_1$ and $u = u_2$ respectively.

We assume that C is composed of a finite number of curves for each of which $r(u)$ has a continuous derivative. Let $A(x,y,z) = A_1 x + A_2 y + A_3 k$ be a vector function of position defined and continuous along C. Then the integral of the tangengential component of A along C from P_1 to P_2, written as

$$\int_{P_1}^{P_2} A \cdot dr = \int_C A \cdot dr = \int_C A_1\,dx + A_2\,dy + A_3\,dz .$$

is an example of line integral. If A is the force F of a particle moving along C, this line integral represents the work done by the force. If C is a simple closed curve, (i.e. a curve which does not intersect itself anywhere, the integral around C is often denoted as :

$$\oint A \cdot dr = \oint A_1\,dx + A_2\,dy + A_3\,dz$$

In aerodynamics and fluid mechanics, this integral is called the circulation of A about C, where A represents the velocity of the fluid.

In general, any integral which is to be evaluated along a curve is called a line integral. Such integrals can be defined in terms of limits of sums as are the integrals of elementary calculus.

Theorem

If $A = \nabla \phi$, everywhere in a region R of space, defined by $a_1 \leq x \leq a_2$, $b_1 \leq y \leq b_2$, $c_1 \leq z \leq c_2$, where $\phi(x,y,z)$ is single-valued and has continuous derivatives in R, then,

1. $\int_{P_1}^{P_2} A \cdot dr$ is independent of the path C in R joining P_1 and P_2.

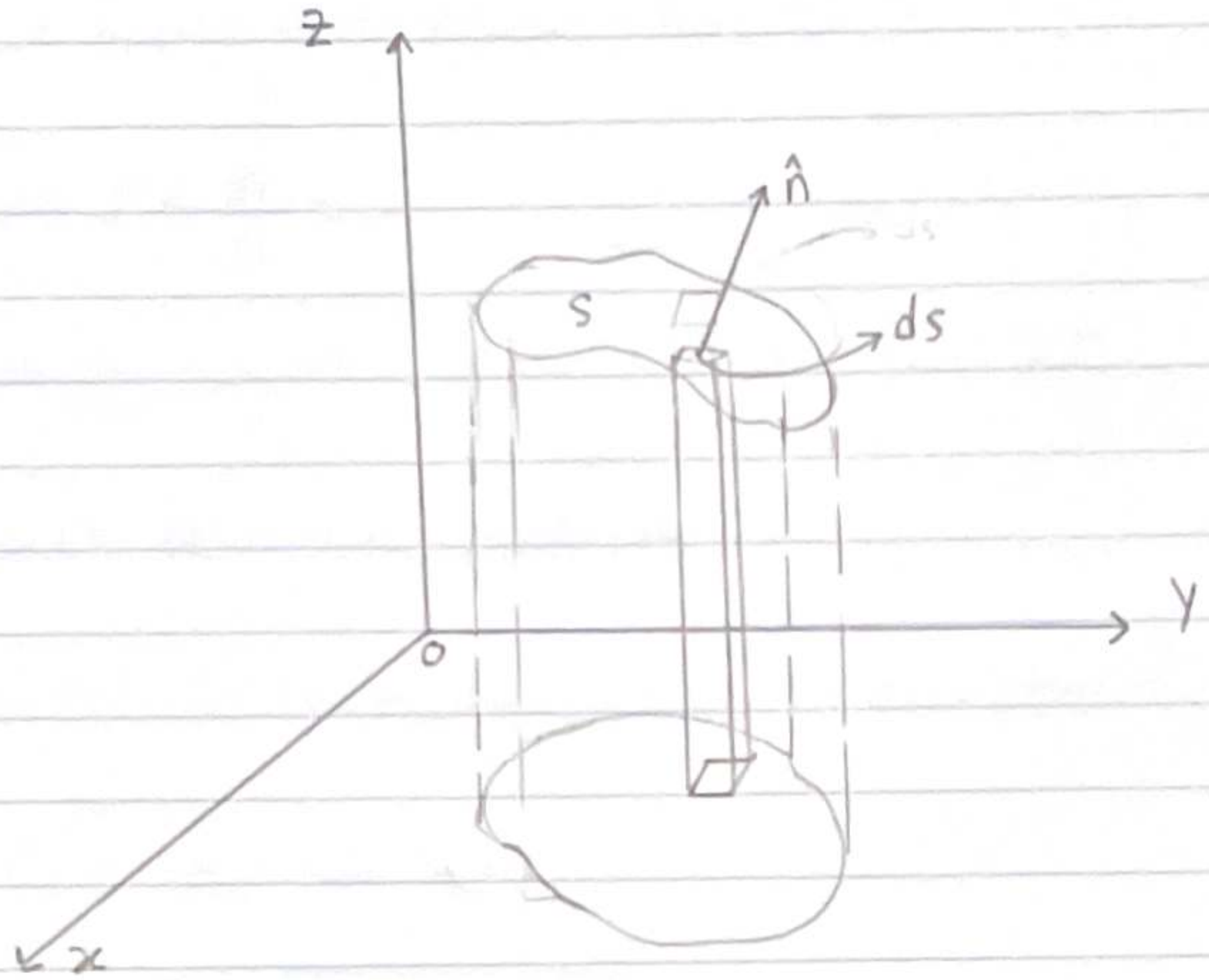
2. $\oint_C A \cdot dr = 0$ around any closed curve C in R.

In such case, A is a conservative vector field and ϕ is a scalar potential.

A vector field A is conservative iff $\nabla \times A = 0$, or equivalently $A = \nabla \phi$. In such case, $A \cdot dr = A_1 dx + A_2 dy + A_3 dz = d\phi$, an exact differential.

Surface Integral

Let S be a two sided surface, such as shown in the figure below. Let one side S be considered arbitrarily as the positive side (if S is a closed surface that is taken as the outer side). A unit normal n to any point of the positive side of S is called a positive or outward drawn unit normal.



Associate with the differential of surface area dS a vector dS whose magnitude is dS and whose direction is that of n. Then dS = n dS.

The integral

$$\iint_S A \cdot dS = \iint_S A \cdot n\, ds$$ is an example of an surface integral called the flux of A over S.

Other surface integrals are :

$$\iint_S \phi\, ds = \iint_S \phi\, n\, ds \ , \ \iint_S A \times dS$$ where ϕ is a scalar function. Such integrals can be defined in terms of ~~limint~~ limits of sums as in elementary calculus.

$\oiint_S$ can also be used to denote surface integral over closed surface.

Volume Integral

Consider a closed surface in space enclosing a volume V. Then

$$\iiint_V A\, dV$$ and $$\iiint_V \phi\, dV$$ are example of volume integrals or space integrals

The equation of motion of a particle P of mass m is given by :

$$m \frac{d^2r}{dt^2} = f(r)\, r_1 \quad -①$$

where r is the position vector of P measured from an origin O. r_1 is a unit vector in direction r and $f(r)$ is a function of the distance P from O.

(a) Show that $r \times \frac{dr}{dt} = c$

(b) Interpret physically the case $f(r) < 0$ and $f(r) > 0$

(c) Interpret the result in (a) geometrically

(d) Describe how the result obtained relate to the motion of planets in ~~the~~ our Solar System.

(a) Multiply ① on both sides by $r \times$

$$m r \times \frac{d^2r}{dt^2} = f(r)\, r \times r_1 = 0$$

Since r_1 is in the direction of r, $r \times r_1 = 0$

$$\Rightarrow m r \times \frac{d^2r}{dt^2} = 0 \qquad f(r)\, r \times r_1 = 0$$

$$\Rightarrow r \times \left(\frac{d^2r}{dt^2}\right) = 0$$

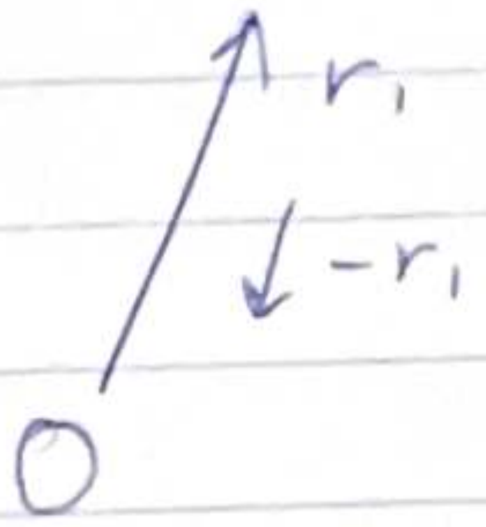
$$\frac{dr}{dt}\left(r \times \frac{dr}{dt}\right) = 0$$

$$\int \frac{d}{dt}\left(r \times \frac{dr}{dt}\right) dt = r \times \frac{dr}{dt} + C$$

$$\Rightarrow r \times \frac{dr}{dt} = 0 \xrightarrow{\text{Intergrating}} \int r \times \frac{dr}{dt}$$

(b) If $f(r) < 0$,

$$m\frac{d^2 r}{dt^2} = f(r)\, r_1$$



$$\downarrow$$

$$r \times \frac{dr}{dt} = c = f(r)\, r_1$$

r is in direction of r_1

if $f(r) < 0 \Rightarrow r \times \frac{dr}{dt} = -r_1$

$\Rightarrow \frac{d^2 r}{dt^2}$ is opposite to r_1 $\Rightarrow$ particle is attracted to O

~~repus~~

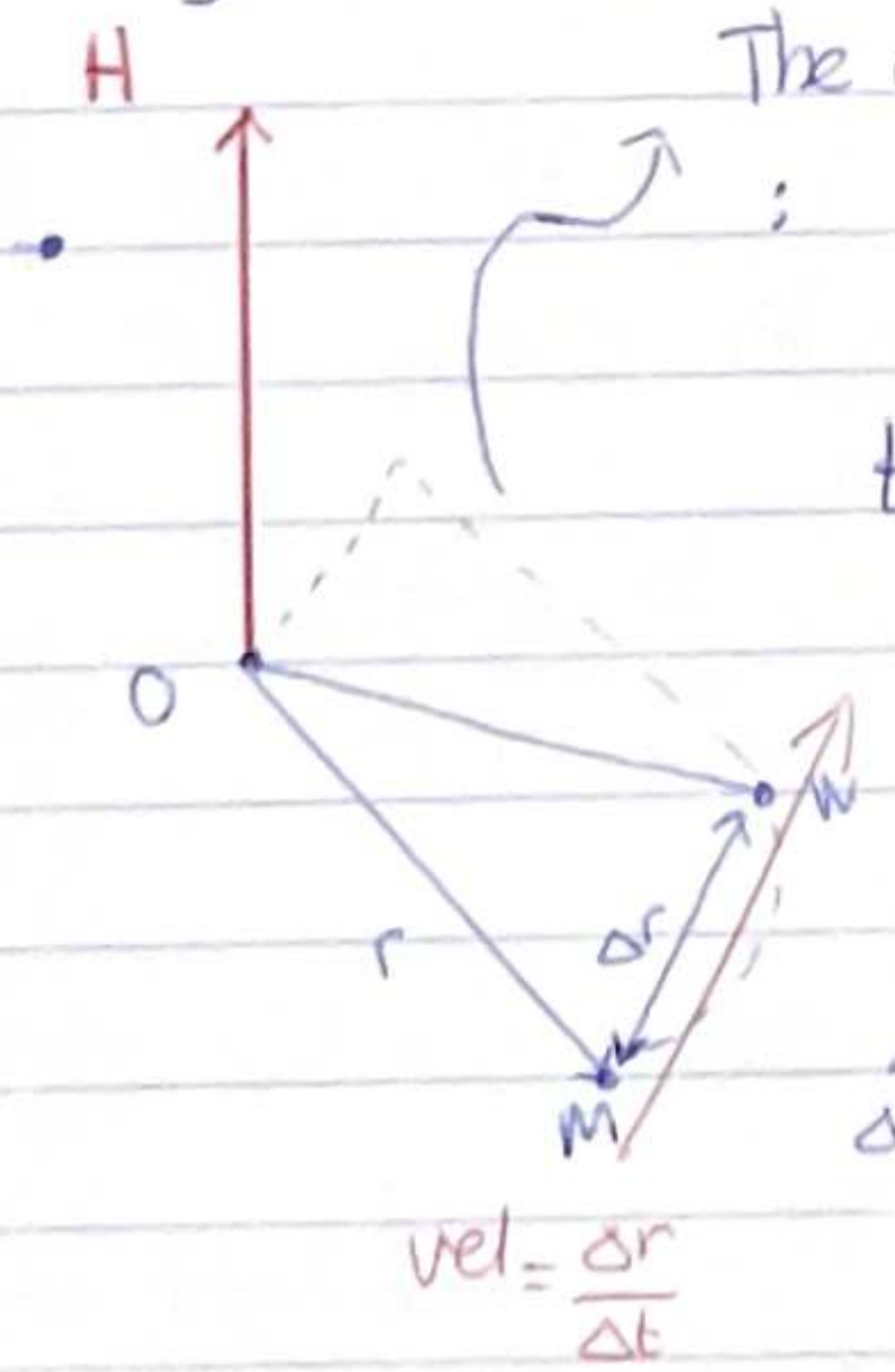
If $f(r) > 0$

$\Rightarrow r \times \frac{dr}{dt} = r_1$

$\frac{d^2 r}{dt^2}$ is towords (in direction or r_1) $\Rightarrow$ repulse from O

$\longrightarrow$ A force directed towords or away from a fixed point with magnitude depending only on r is called a Central force.

(c) Consider particle moving from M to N in time interval Δt



The area swept in the interval Δt is approximately

: $\frac{r \times \Delta r}{2} = \frac{1}{2} r \times \Delta r$

this area was swept by r in Δt / unit time

$= \frac{1}{2} r \times \frac{\Delta r}{\Delta t}$

$\Rightarrow$ instenteneous rate of change in area is

$$\lim_{\Delta t \to 0} = \frac{1}{2} r \frac{\Delta r}{\Delta t} = \frac{1}{2} r \frac{dr}{dt}$$

$$= \frac{1}{2} r V$$

instenteneous velocity of the particle.

H is a vector which is orthogonal to the plane where r & $\dot{r}$ lie.

$$\Rightarrow \underbrace{H}_{\text{Areal Velocity}} = \frac{1}{2} r \times \frac{dr}{dt} \quad = \quad H = \frac{1}{2} r \times V$$

from (iv) $\quad r \times \frac{dr}{dt} = \text{Constant} \quad \Rightarrow H = \frac{1}{2} r \times \frac{dr}{dt} = \text{Constant}$

Since $H \cdot r = 0 \quad \Rightarrow$ Motion takes place only in plane where r & $\dot{r}$ are present.

(d) A planet (Such as Earth) is attracted towards the Sun according to Newton's Universal law of gravitation, which states that any two object of mass m and M respectively are attracted towards each other with a force $F = \frac{GMm}{r^2}$ where r is the distance between the 2 objects and G = Universal grav constant.

Let m and M be the mass of planet & Sun respectively with origin O at the Sun. The equations of motion of the planet is

$$m \frac{d^2 r}{dt^2} = -\frac{GMm}{r^2} r_1 \quad \text{or} \quad \underbrace{\frac{d^2 r}{dt^2}}_{\frac{dv}{dt}} = -\frac{GM}{r^2} r_1$$

assuming the influence of other planets is negligible.

according to part (c) the Planet ~~swee~~ moves around the Sun ~~and sweeps~~ its position vector sweeps out equal areas in equal time

5

Show that the path of a planet around the Sun is an ellipse with the Sun at one focus.

From 4(c) & 4(d)

$$\cancel{m\frac{d^2r}{dt^2} = \frac{-GMm}{r^2}}$$

$$\frac{d^2r}{dt^2} = \frac{dv}{dt} = -\frac{GM}{r^2} r_1 \quad -①$$

$$\frac{1}{2} r \times v = H$$

$$\Rightarrow r \times v = 2H = h$$

differentiate

$$\underbrace{r = r r_1}_{\text{product rule}} \Rightarrow \frac{dr}{dt} = \cancel{r\frac{dr_1}{dt} r_1}\ r\frac{dr_1}{dt} + \frac{dr}{dt} r_1$$

$$h = r \times v$$

$$\Rightarrow h = r r_1 \times \left(r\frac{dr_1}{dt} + \frac{dr}{dt} r_1\right) = r^2 r_1 \times \frac{dr_1}{dt}$$

From ①
$$\frac{dv}{dt} \times h = -GM r_1 \times \left(r_1 \times \frac{dr_1}{dt}\right)$$

$$= -GM\left[\left(r_1 \cdot \frac{dr_1}{dt}\right) r_1 - (r_1 \cdot r_1)\frac{dr_1}{dt}\right] = GM\frac{dr_1}{dt} \quad ③$$

Using ③ and the fact that $r_1 \cdot \frac{dr_1}{dt} = 0$

But, Since h is a constant vector : $\frac{dv}{dt} \times h = \frac{d}{dt}(v \times h)$ so that

$$\frac{d}{dt}(v \times h) = GM\frac{dr_1}{dt}$$

Integrating both sides

$$v \times h = GM r_1 + P$$

$$r\cdot(v\times h) = GM\, r\cdot r_1 + r\cdot p$$

$$= GMr + \underbrace{r\, r_1\cdot p}_{\text{dot product}}$$

$$= GMr + r\,r\,p\cos\theta$$

Where p is a constant vector with magnitude p and θ is the angle between p and r_1

Since $r\cdot(v\times h) = (r\times v)\cdot h = h\cdot h = h^2$

We have $h^2 = GMr + rp\cos\theta$ and

$$r = \frac{h^2}{GM + p\cos\theta} = \frac{h^2/GM}{1+(p/GM)\cos\theta} \quad \text{— (4)}$$

Argument from Geometry (polar coords. of Conic Sections)

$\Rightarrow$ Polar equation of a conic section with focus at the origin and eccentricity ε is

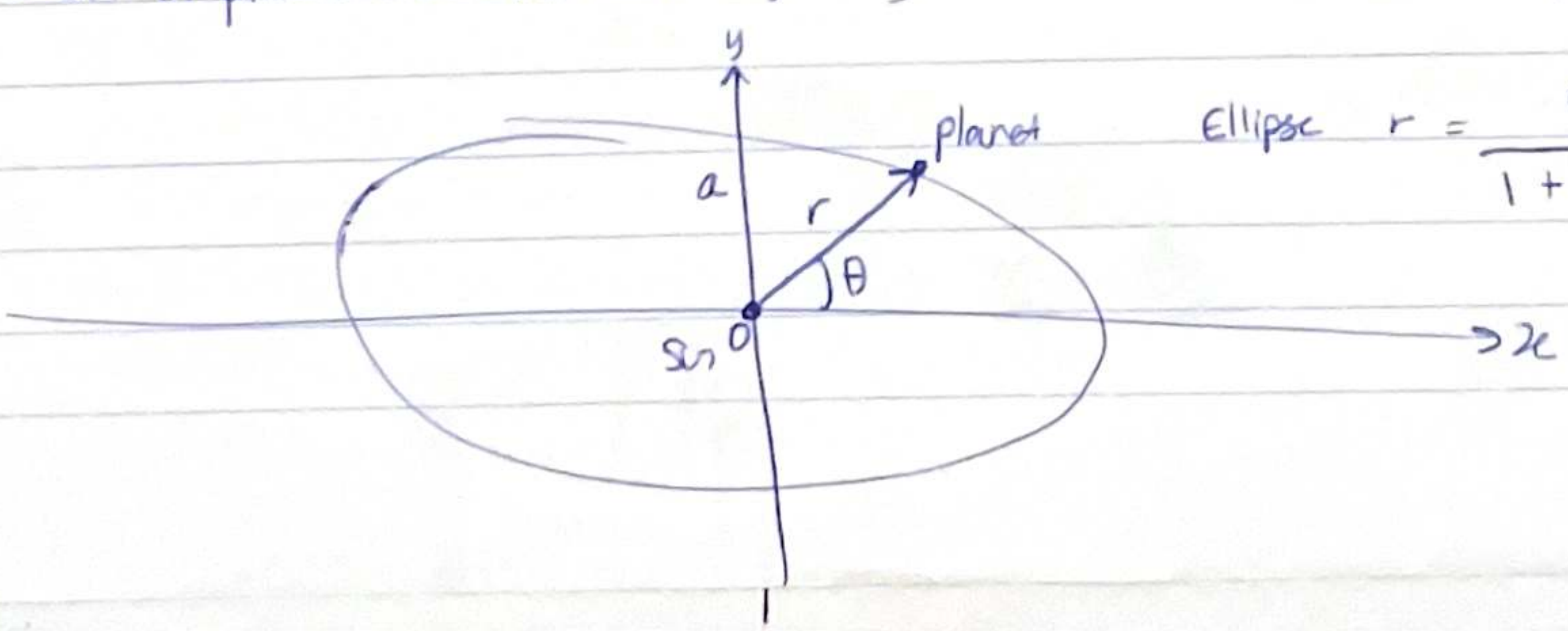
$$r = \frac{a}{1+\varepsilon\cos\theta}$$ where a is a constant, Comparing with (4)

It can be concluded the the required orbit is a conic section with eccentricity $\varepsilon = P/GM$

The orbit can be an ellipse, parabola or hyperbola as ε is less than, equal to or greater ~~greater or equal~~ than one

$\Rightarrow$ no circle $\hookleftarrow$ * circle has no eccentricity.

Since orbit of planets are closed curve, $\Rightarrow$ Orbit must be an ellipse.



Ellipse $r = \dfrac{a}{1+\varepsilon\cos\theta}$

Product Identities

Dot product is Commutative, Cross product is not.

d/dx is distributive.

Dot product is distributive

① $A \cdot A = A^2 = A_1A_1 + A_2A_2 + A_3A_3$

② $A \cdot B = A_1B_1 + A_2B_2 + A_3B_3$

$A \cdot B = B \cdot A$
$A \cdot (B + C) = A \cdot B + A \cdot C$
$m(A \cdot B) = (mA) \cdot B = (mB) \cdot A = (A \cdot B)m$
$\hat{i} \cdot \hat{i} = 1 \quad \hat{i} \cdot \hat{j} = 0$

If $A \cdot B = 0 \Rightarrow$ A & B are perpendicular

$A \times B = -B \times A$

$A \times (B + C) = A \times B + A \times C$

$m(A \times B) = (mA) \times B = A \times (mB) = (A \times B)m$

$\hat{i} \times \hat{i} = 0 \qquad \hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i} \quad \hat{k} \times \hat{i} = \hat{j}$

If $A \times B = 0 \Rightarrow$ A & B are parallel

$$A \times B = \begin{array}{c} \\ \end{array}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Triple product

1. $(A \cdot B)C \neq A(B \cdot C)$

2. $A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B) \equiv A \cdot B \times C$

3. $A \times (B \times C) \neq (A \times B) \times C$

4. $A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$

$(A \times B) \times C = (A \cdot C)B - (B \cdot C)A$

dient ∇

(x,y,z) be differentiable at each point (x,y,z) in a certain region of space.

$$\nabla\phi = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

The component of $\nabla\phi$ in the direction of unit vector a is given by $\nabla\phi \cdot a$. It is the rate of change of ϕ at (x,y,z) in the direction of a.

Divergence $\nabla \cdot V$

$V(x,y,z) = V_1 i + V_2 j + V_3 k$

V defines a differentiable vector field.

$$\nabla \cdot V = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left(V_1 i + V_2 j + V_3 k\right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

$$\nabla \cdot V \neq V \cdot \nabla$$

Curl $\nabla \times V$

$V(x,y,z)$ is a differentiable vector field, then Curl of V is defined as

$$\nabla \times V = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \times \left(V_1 i + V_2 j + V_3 k\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_2 & V_3 \end{vmatrix} i - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_1 & V_3 \end{vmatrix} j + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_1 & V_2 \end{vmatrix} k$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) i + \left(\frac{\partial V_1}{\partial x} - \frac{\partial V_3}{\partial z}\right) j + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) k$$

If $\phi(x,y,z) = 3x^2y - y^3z^2$

find $\nabla\phi$ at $(1,-2,-1)$

$$\nabla\phi = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$

$$= i\frac{\partial(3x^2y - y^3z^2)}{\partial x} + j\frac{\partial(3x^2y - y^3z^2)}{\partial y} + k\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= 6xy\, i + 3x^2 - 3y^2z^2\, j + - 2y^3z\, K$$

$$= 6(1)(-2)i + 3(1)^2 - 3(-2)^2(-1)^2 j - 2(-2)^3(-1) K$$

$$= -12i - 9j - 16K$$

Vector Differentiation

$$\frac{d}{du}(A+B) = \frac{dA}{du} + \frac{dB}{du}$$

$$\frac{d}{du}(A.B) = A\frac{dB}{du} + B\frac{dA}{du}$$

$$\frac{d}{du}(A \times B) = A \times \frac{dB}{du} + B \times \frac{dA}{du}$$

$$\frac{d}{du}(\phi A) = \phi\frac{dA}{du} + \frac{d\phi}{du}A$$

$$\frac{d}{du}(A.B \times C) = A.B \times \frac{dC}{du} + A\frac{dB}{du} \times C + \frac{dA}{du}.B \times C$$

$$\frac{d}{du}\left[A \times (B \times C)\right] = A \times \left(B \times \frac{dC}{du}\right) + A \times \left(\frac{dB}{du} \times C\right) + \frac{dA}{du} \times (B \times C)$$

Partial derivative

$$\frac{\partial A}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial A}{\partial x}\right), \quad \frac{\partial^2 A}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial A}{\partial y}\right)$$

$$\frac{\partial}{\partial x}(A.B) = A.\frac{\partial B}{\partial x} + B.\frac{\partial A}{\partial x}$$

$$\frac{\partial}{\partial x}(A \times B) = A \times \frac{\partial B}{\partial x} + B \times \frac{\partial A}{\partial x}$$

$$\frac{\partial^2}{\partial y \partial x}(A+B) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(A.B)\right) = \frac{\partial}{\partial y}\left(A.\frac{\partial B}{\partial x} + B.\frac{\partial A}{\partial x}\right)$$

$$= A.\frac{\partial^2 B}{\partial x \partial y} + \frac{\partial A}{\partial y}.\frac{\partial B}{\partial x} + \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial^2 A}{\partial x \partial y}.B$$

Combine all forces in a matrix.

$$P = \begin{vmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{vmatrix}$$

P_{xx}, P_{yx}, P_{zx} Cannot be added to form a vector although they are in the same direction as nature of forces is different.

P_{xx} acts to 'pull' O forward

P_{yx} acts to shear O

Stress can be specified

each component of P corresponds to a force per unit area on a particular surface where O is present and direction along which it acts

P has 9 components (2 basis vector / component

1 magnitude of force

1 for direction of force

(Area & force in line)

2 $\hat{i}$ vectors in x direction

1 vector $\hat{i}$ in x direction (Area)

1 vector $\hat{j}$ in y direction. (force)

$$P = \begin{vmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{vmatrix}$$

Rank 2

P_{xz} → direction of force.

↓

direction Area is ⊥ to

Components : m^n : $3^2 = 9$

Tensor : in an m-dimensional space, a tensor of rank n is a mathematical object that has n indices, m^n components and it obeys certain transformation rule

From example : above $m = 3$

Rank of tensor : No of basis vector needed to fully specify components

↑ (n)