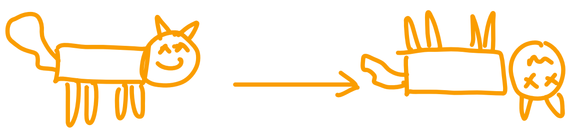
Extinction of the Human Race

Extinction is defined as the loss of a species. Throughout the history of our planet, various life forms previously present now only remain as bones! From the Saber-tooth tiger which existed 2.5 million years ago to 10 000 (ya) to the Splendid poison frog declared extinct in 2020, that cruel process of life is a looming threat to fauna and flora. One question which remains at the back of the mind is the methods of extinction of Homo Sapiens.



To better grasp the concept, let's take a closer look at the dynamics of a biological System. In biomathematics, predator-prey interaction and outcomes are modelled by a set of non-linear ODEs known as the LOTKA-VOLTERRA EQUATIONS.

Consider a simple model where x is the no of zebras (prey) and y the no of tigers (predators).

$$\frac{dx}{dt} = \alpha x - \beta xy$$

$$\frac{dy}{dt} = \delta xy - \gamma y$$

Without Solving,

From the 1st eqn we can see that if the predators vanish ($y=0$) we would obtain exponential growth of preys which is consistent with the law of growth.

Similarly for the 2nd eqn, removing preys result in predator exponential decay.

In the 1st eqn we note the term $-\beta xy$ and see that y has a -ve interaction with x i.e. when x AND y meet, the growth of x (prey) is reduced. Similarly for the term δxy in the 2nd eqn, x has a +ve interaction i.e. x AND y meeting $\Rightarrow$ y growth.

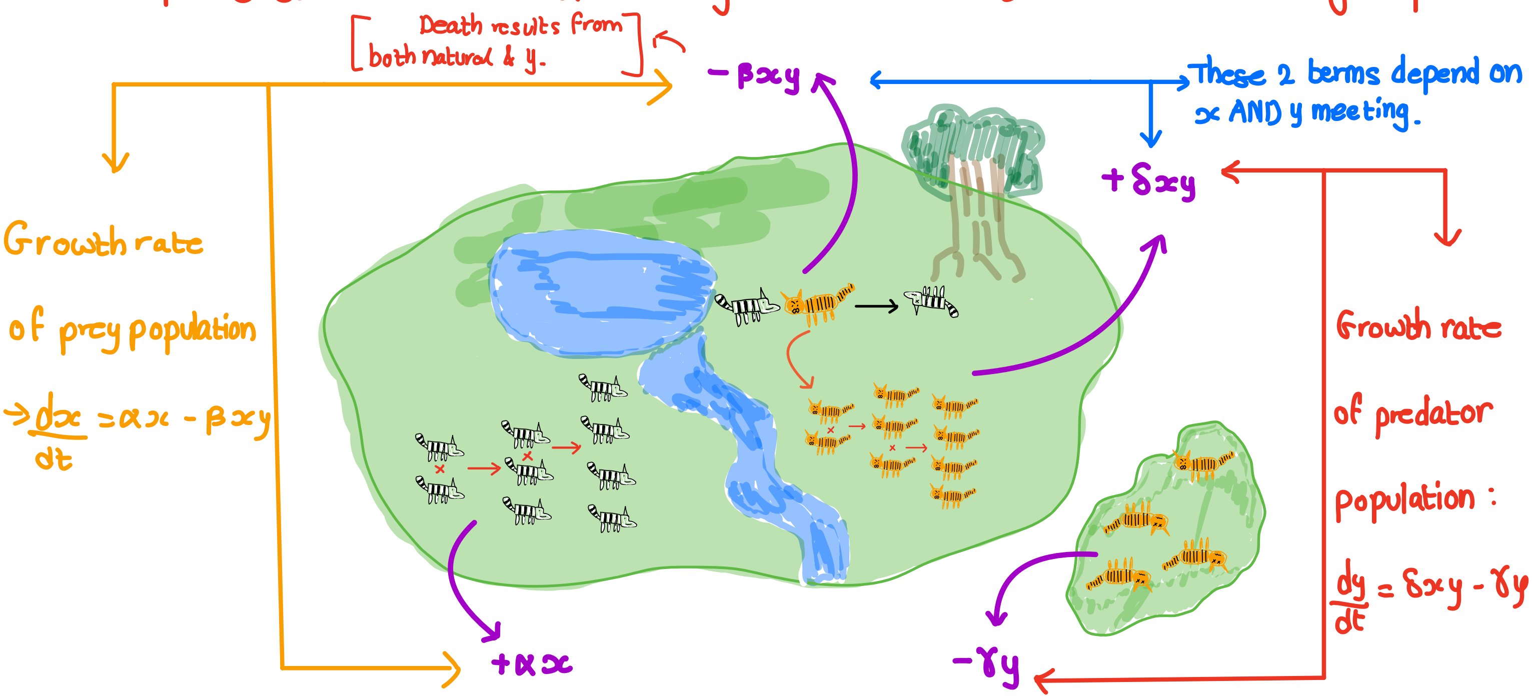
The green letters are the real coefficients

or "dumbness" to get eaten!

- α directly influences growth and could be a fitness/fertility factor for preys. γ influences death of predators and could be related to life span.

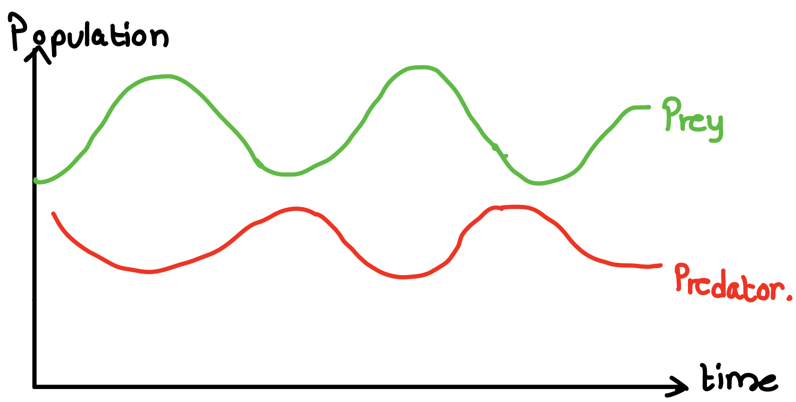
or inter predator competition.

- β from the rate of predation could represent the speed at which a predator devours a prey. δ relates to growth of predators and could be a factor of predation efficiency + fitness. (Absolute fitness) $\longrightarrow$ In biology fitness measures the "reproductive success" of an organism and its contribution to the gene pool.



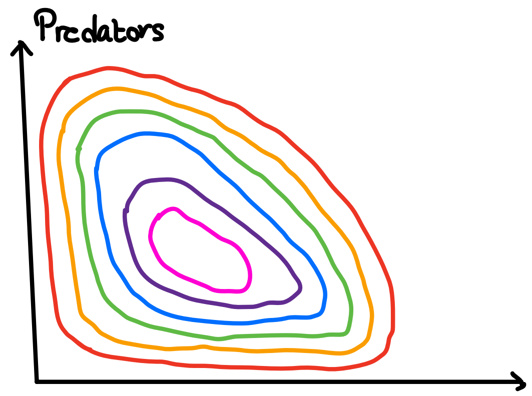
Solving will lead to a periodic soln not expressible in the usual trigonometric power series.

The equations can be linearized and a plot similar to this is obtained.



$\longleftarrow$ Represent both individual and mutual interactions.

- We can see that predator population lags prey by 90°
- Also predator population is always less than prey.



Each colour represents a set of initial population values (x,y).

The orbits of the phase plot represent the variation (cyclic) of prey-predator population.

We also notice that the orbits are attracted to a non-zero fixed point. That singularity represents the values of (x,y) for which population size remain constant! We also notice that the fixed-point is non-zero. ⇒ EXTINCTION IS NOT POSSIBLE for the chosen green letters parameters.

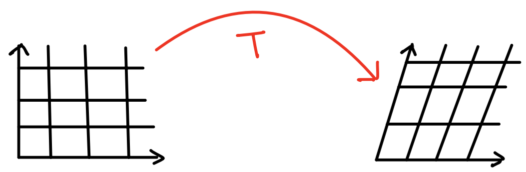
$$\alpha \ln y + \gamma \ln x = \delta x + \beta y + K$$

$$y^{\alpha} x^{\gamma} = e^{\delta x + \beta y + K}$$

$$y^{\alpha} = \frac{e^{\delta x + \beta y + K}}{x^{\gamma}}$$

Stability Analysis of the Fixed Point.

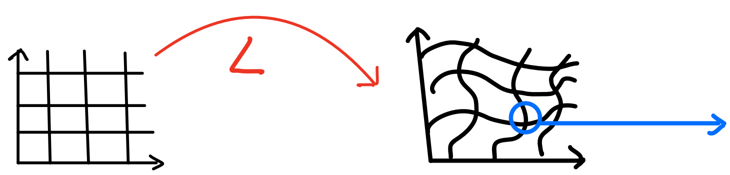
To do so, we shall borrow some stuff from Linear Algebra, namely the determinant.



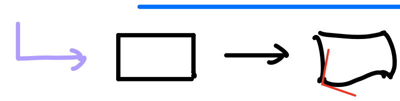
The determinant encodes useful information on how T morphs $\mathbb{R}^2$. i.e. $T : (x,y) \mapsto (X,Y)$

We are doing so because a system of ODE can be expressed as vectors and thus vector algebra can be used! But we encounter a problem, our system of ODE is non-linear.

Non-linear transformations look like this.



If we look very closely at small scales, we can see that L behaves linearly.



The equations defining T takes a set of values and send it somewhere else. T is a linear function.

The ODEs defining L takes a set on initial population values and send it to another value.

We are trying to find the stable point to which L can send the values and determine the nature of that stable point.

Concerning the ODE partial derivatives can be used to see how x behaves without the influence of y and same for y. Setting this up in the usual vector notations of Linear Algebra gives us the

JACOBIAN MATRIX.

Community Matrix.

$$J(x,y) = \begin{bmatrix} \frac{\partial f_1}{x} & \frac{\partial f_1}{y} \\ \frac{\partial f_2}{x} & \frac{\partial f_2}{y} \end{bmatrix} \rightarrow \begin{array}{l} \frac{dx}{dt} = \alpha x - \beta x y \\ \frac{dy}{dt} = \delta x y - \gamma y \end{array} \rightarrow \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - y \end{bmatrix}$$

↑ Treat y as constant.

Population Equilibrium.

Equilibrium occurs when neither population sizes are changing

$\Rightarrow$ at $\frac{dx}{dt} = 0 \quad \Rightarrow \alpha x - \beta x y = x(\alpha - \beta y) = 0$

$\frac{dy}{dt} = 0 \quad \Rightarrow \delta x y - \gamma y = y(\delta x - \gamma) = 0$

Soln: $(x,y) = (0,0)$ ✗ — Extinction of both species — We have seen that the fixed point is non-zero.

$(x,y) = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$

At these values of the greeks, population size is constant

↳ Death is followed by immediate birth!

Stability

At $(0,0)$

Community matrix is : $J(0,0) = \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \end{bmatrix}$

$$|J(0,0) - \lambda I| = \begin{bmatrix} \alpha - \lambda & 0 \\ 0 & -\gamma - \lambda \end{bmatrix}$$

Characteristic Polynomial : $(\alpha - \lambda)(-\gamma - \lambda) = 0$

$\lambda_1 = \alpha \quad \lambda_2 = -\gamma$

↗ By invertible Matrix theorem non-trivial soln for invertible Matrix only when det. = 0

why ??

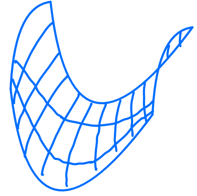
(Zoom further)

Remember α & γ are fertility factors and must be greater than 0. $\Rightarrow$ The signs differ always.

$\Rightarrow$ Saddle Point. at $(0,0)$ $\longrightarrow$ A saddle point is a fixed point which is both at its minimum in one variable and maximum at the other.

Imagine a horse saddle.

The pink fixed point is minimum for the red axis but maximum for the blue axis and might "fall off" left or right $\Rightarrow$ unstable!



The origin being unstable fixed point is a good thing! It means that extinction is difficult!

The population of prey and predator can get close to 0 and still recover.

Second fixed Point (Oscillations)

$$J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{bmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\beta} & 0 \end{bmatrix}$$

What do the eigenvalues represent?

$$|J-\lambda I| \Rightarrow \lambda_1 = i\sqrt{\alpha\gamma} \quad \lambda_2 = -i\sqrt{\alpha\gamma}$$

$\leftarrow$ The imaginary values have opposite signs. (Conjugate).

$$(J-\lambda I) = \begin{bmatrix} -\lambda & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\beta} & -\lambda \end{bmatrix}$$

$$= \lambda^2 + \alpha\gamma$$

$$= \lambda_{1,2} = \pm i\sqrt{\alpha\gamma}$$

$\Upsilon = 0$

Evaluation of Stability.

We have No change in population size at $(0,0)$ and $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$

At $(0,0)$

$$J(0,0) = \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \end{bmatrix} \Rightarrow |J(0,0) - \lambda I| = (\alpha-\lambda)(-\gamma-\lambda) = 0$$

$$\Rightarrow \lambda_{1,2} = \alpha, -\gamma$$

(Using vieta's formula

$$\delta = -\alpha\gamma$$

$$\Upsilon = (\alpha - \gamma)$$

$$\Rightarrow |J(0,0) - \lambda I| = \lambda^2 + (\gamma - \alpha)\lambda - \alpha\gamma = 0$$

$$\Rightarrow \lambda = \frac{(\alpha-\lambda) \pm \sqrt{(\gamma-\alpha)^2 + 4\alpha\gamma}}{2}$$

Since the greeks were defined as +ve reals, $\Rightarrow$ They always differ in sign.

$$\Rightarrow \alpha > -\gamma \quad \Rightarrow \delta < 0$$

$$\Rightarrow \boxed{\Upsilon \text{ Can Vary Sign!}}$$

α	γ
2	5
3	3
5	2

$\Rightarrow$ Hyperbolic Point.

Why?

We have $\lambda = \frac{\Upsilon}{2} \pm \sqrt{\frac{\Upsilon^2}{4} + |\delta|}$

$\Rightarrow \lambda$ always real!

$\Rightarrow$ More on mobius Transformations & Stability Analysis!