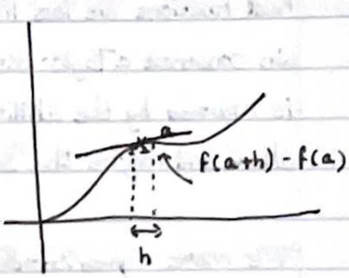
Differentiation

The first derivative of f at a,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



This gives us f'(a) which are coefficients for our power Series

We notice that $\frac{d}{dx}(x-a)^n = n(x-a)^{n-1}$

$$\Rightarrow \frac{d^n}{dx^n}(x-a)^n = n!$$

To recover purely the contribution of the coefficients near a without multiplicative differentiation amplification, we exclude the factorial term $\Rightarrow \frac{1}{n!} \times f^n(a)$

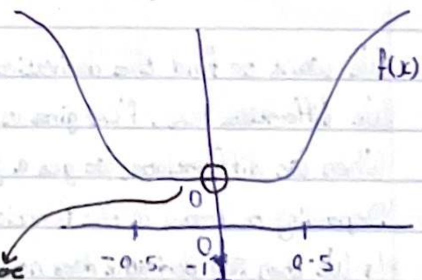
Now we have coefficients from taking derivatives.
~~Will the coefficients lead to the convergence of the Series to f?~~

This function is continuous, Smooth, but not analytic.

Real Analysis & Complex Analysis.

Example, Smooth & non - Analytic function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$



Derivatives exist everywhere
But taylor Series only converges at 0
it fails to converge anywhere else.

Analytic Continuation is not guaranteed.

The thing about ~~real~~ analysis is that it sees a function piece by piece. In the real case, the function is modular. It might be Smooth & Analytic on one interval and fail in the other.
The Real case has no pattern to ensure the next piece will be well behaved.
Complex analysis is more rigid. It is like a fractal. If it is well behaved for one Piece, then it is well behaved on the whole domain. Analytic continuation is more likely.

Real Analytic Functions & Complex analytic functions

Functions of Real Variables can or Cannot be analytic
This depends on the function.
For the Real Case, it has nothing to do with a function being Smooth (differentiability)

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Smooth but not Analytic.

function does not equal taylor-series for points $x \neq 0$

Complex Smoothness is more restrictive.

A non-continuous function is not differentiable at Point of break thus cannot be smooth on the interval.

$\boxed{\text{Conditions : Differentiable , Analytic, Continuous , Smoothness}}$

For the real case, we need a function that is
i) not Smooth
ii) Smooth & Analytic
iii) Smooth & non-Analytic.

Suppose we have a function f and we are interested at point a.
The function needs to be continuous at that point. (No gaps)
The function must ~~be~~ have first order derivative [differentiable]
The function must be infinitely differentiable [Smooth]
The Taylor Series must converge to the function [Analytic]
A function can also be smooth & non-differentiable.

Smoothness does not imply analyticity in the real domain but it does in the Complex one.
This hinges on differentiability in each.

Some how, despite being infinitely differentiable, f can escape analyticity in $\mathbb{R}$ but not in $\mathbb{C}$
Graphically, how does differentiation allow power Series to escape in $\mathbb{R}$ but not in $\mathbb{C}$.

What is differentiation & how does power Series relate to it?

Analytic Continuation (Extension of the domain of an analytic function)

Consider a particular analytic function f. It is given by ~~k~~ a power series centred at $z=1$.

$$f(z) = \sum_{k=0}^{\infty} (-1)^k (z-1)^k$$

By the Cauchy-Hadamard Theorem, its radius of convergence is 1. That is f is defined and analytic on the open set $U = \{|z-1|<1\}$ which has boundary $\partial U = \{|z-1| = 1\}$. Indeed, the series diverges at $z=0 \in \partial U$.

Let us re centre the power series at a different point $a \in U$

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k.$$

We will calculate the a_ks and determine whether this new power series converges in an open set V which is not contained in U. ~~So~~ If so, we will have analytically continued f to the region $U \cup V$ which is strictly larger than U.

The distance a to ∂U is $\rho = 1 - |a-1| > 0$. Take $0<r<\rho$; let D be disk of radius r around a, and let ∂D be its boundary.

Then $D \cup \partial D \subset U$

(But what to do when at singularities eg 0 for $\frac{1}{x}$?)

(Analytic continuation for real valued functions.)

Using Cauchy's differentiation formula to calculate the new coefficients:

$$a_k = \frac{f^{(k)}(a)}{k!} = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{k+1}} = \frac{1}{2\pi i}\int_{\partial D} \frac{\sum_{n=0}^{\infty}(-1)^n(\zeta-1)^n\,d\zeta}{(\zeta-a)^{k+1}}$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(-1)^n\int_{\partial D}\frac{(\zeta-1)^n\,d\zeta}{(\zeta-a)^{k+1}} = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(-1)^n\int_0^{2\pi}\frac{(a+re^{i\theta}-1)^n\,rie^{i\theta}\,d\theta}{(re^{i\theta})^{k+1}}$$

$$= \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\int_0^{2\pi}\frac{(a-1+re^{i\theta})^n\,d\theta}{(re^{i\theta})^k}$$

$$= \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\int_0^{2\pi}\frac{\sum_{m=0}^{n}\binom{n}{m}(a-1)^{n-m}(re^{i\theta})^m}{(re^{i\theta})^k}$$

$\vdots$

$$= (-1)^k a^{-k-1}$$

That is: $f(z) = \sum_{k=0}^{\infty} a_k(z-a)^k = \sum_{k=0}^{\infty}(-1)^k a^{-k-1}(z-a)^k = \frac{1}{a}\sum_{k=0}^{\infty}\left(1-\frac{z}{a}\right)^k$

which has radius of convergence $|a|$, and $V = \{|z-a|<|a|\}$.

Analytic Continuation (Extension of the domain of an analytic function).

Consider a particular analytic function f. It is given by ~~k~~ a power Series centered at $z = 1$.

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By the Cauchy - Hadamard Theorem, its radius of convergence is 1. That is f is defined and analytic on the open set $U = \{ |z-1| < 1 \}$ which has boundary $\partial U = \{ |z-1| = 1 \}$. Indeed, the series diverges at $z = 0 \in \partial U$.

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