Dual Space V*

We have a vector space V in which all elements of V respect the Vector Axioms.

We have a linear map T: V → W that maps elements of one vector space to another.
It is a function restricted to maintain the vector axioms for the elements, and thus preserving vector space structure.

What is a linear map?

We have an ordinary function f : X → Y
It is a relationship between the elements of the domain X and the elements of the co-domain Y, an elementwise rule.

Homomorphisms

It is a map between 2 algebraic structures that preserves the operations defining the structure. It is a natural map as it does not destroy the structure of the objects. We have morphisms and categories:

- Continuous maps for Topological Spaces
- Group Homomorphisms
- Ring Homomorphisms
- Linear Maps for Vector Spaces
- Functions for Sets

[Sets are collection of objects without any structure; so, morphism trivially has no structure to preserve apart from the mapping rule]

Compared to functions and sets which are just connecting one element to another, with underlying structures and the requirement of its preservation, the domain and range are just instances of the same structure, and the morphism effectively morphs the Dom to the Cdm .

Different instances mean different elements, but the same structure is shared; Like R^2 and R^3.
The mapping of elements is maintained while respecting the structure's allowed transformations.
We have an algebraic structure and its elements making up the vector space. Two vector spaces V and W share the same underlying structure. They differ in instances by the elements they possess.

The elements in the space are not just junk data; they are closed under a way that respects vector axioms. For example, the elements of R^3 and the set of polynomials with degree less or equal to 2. The sets look different owing to different domains, but they share the same blueprint.

A morphism exists because the elements are tied to the rules of the vector space and the morphisms represents a translation from one realization of V to W. Whatever role an element plays in V, the morphism points to that role in W.

Example, if we embed R^2 as a slice of R^3, the slice still behaves similarly with all coordinates in form (x, y, 0).

<u>Linear Functionals</u>
Linear functional/Linear form/One-form is a map from a vector space V to its underlying field F. It does so by collapsing a vector into a scalar in a way that respects vector space structure.

They extract information from a vector.

In physics, linear functionals are observable, Work = Force . displacement.

Displacement is the vector and fixed Force component behaves as the linear functional.

In Quantum Mechanics, states live in a vector space; linear functionals (bra vectors) extract amplitude (observables) from state vectors (ket vectors).

The dot product/inner product and integration are common linear functionals.

One linear functional can generate an output when fixing a vector used for the dot product or when fixing a weighed integral.
However, the collection of all linear functionals, not their outputs is hom(V, F); adding linearity in addition and scaling forms the dual space V* of V.

A linear functional/one-form is also called a co-vector.

One co-vector gives us one number; but all co-vectors together give us all the ways of measuring a vector.

Two vectors giving the same result under all co-vectors implies that they are the same vector.

V is the set of objects in the world and V* is the set of measuring tools.

Also include part where any element of V can be used to evaluate other via dot product or any f can be used to weigh other via integration, inner product, and integration connection
Also mention change of basis and arbitrariness of understanding the system

Basis and Dual Basis

Suppose we have a subspace S of V. S contains some elements. Under Linear Combinations, other elements of V can be reached. The span of S is the subspace of V such that it contains the elements of S and all its possible linear combinations; span (S).

Span (S) cannot contain any element outside V because S is defined as a subspace of V and a subspace respects vector axioms enforcing closure under scalar multiplication and addition. The axioms are necessary and sufficient to present linear combination as an operation on elements of vector space.

A basis B is a special kind of subspace B such that span (B) reaches the whole vector space V.
Also, the elements of B are linearly independent; that is, one cannot be expressed as a linear combination of the other.

This prevents redundancy of the minimal basis set. Also, neutral elements which do nothing are excluded. If we take the zero vector, any coefficients combined with the zero vector will yield 0. Usually linear independence requires all coefficients (from the field) to be 0 to yield the 0 vector. Here, we are getting nontrivial 0 vectors.

<u>The Basis in Relation to Linear Maps</u>
We know vector spaces are well defined and ruled algebraic structures. The elements living in there respect these rules. We also know linear maps are functions relabelling the elements when translating between vector spaces.

We can interpret what we said geometrically. The well ruled vector space is represented by a grid structure. In one space V, one grid equals one unit. In the other space W, one grid equals 2 units. The linear transformation from V to W, while still respecting the grid structure can be seen as squishing the line in V to ½ its length in W. Keeping both grids the same across V and W, we see that 1 was sent to ½ .

From the importance of bases for a space, the image of bases under a linear map is something important. It gives us a way to know exactly where goes where for any possible linear combinations in the domain.

Dual Basis

The word co- in mathematics is interesting. It means something opposite to its suffix. We have sets which are an unordered collection of elements without any particular order or interest.

A coset induces partition rigidly onto the elements connected only by logic; a complete atomic Boolean algebra is the opposite of a set with join and meet operations between individual elements of the set.

Duals are also known as co-vectors, they are the opposite of a vector; while a vector is an element of a vector space, a dual represents a linear functional; a number associated with that vector.

Together, the co-vector and the vector, uniting both opposites generate a complete image of what we are observing, the set with all the unordered objects and the coset, showing the distinction of elements and the logic between them.

The basis being the generator set of the vector space; the dual basis should correspond to something as important; it could be a sort of parent function guiding all the linear functionals.

The dual basis serves as parent for all linear functionals; if the basis is the coordinate generating the set, the dual basis is the function extracting the number/scalar of each coordinate.

The dual basis essentially says this is this; this is something fixed which I measure and define, and all other measurements will follow that.

Naturally, under a change of basis, a dual basis will change accordingly

V* Forms a Vector Space

The dual basis plays the role of building blocks for all linear functionals much like basis for vectors.

Every linear functional of V* can be expressed as the linear combination of elements of the dual basis set.

If V is an n-dimensional vector space with basis $B = \{v_1, \ldots, v_n\}$, then the dual basis $B^* = \{\varphi_1, \ldots, \varphi_n\}$ has this key property:

Every linear functional $f \in V^*$ can be **uniquely expressed** as:

$$f = a_1\varphi_1 + a_2\varphi_2 + \cdots + a_n\varphi_n,$$

a is element of underlying Field. All functionals can be expressed as LC of basis coordinate extractor

2. Coordinate Extraction Role

Given a vector $v = b_1v_1 + \cdots + b_nv_n$,

$$\varphi_i(v) = b_i$$

so φ_i is literally the "coordinate-extraction functional" for the i-th coordinate.
Any other functional is just some weighted sum of these coordinate extractions.

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3. Analogy

Vectors	Linear Functionals
Basis vectors v_i span V	Dual basis functionals φ_i span V^*
Any vector is a linear combination of v_i	Any functional is a linear combination of φ_i
Basis is "parent vectors" that generate all others	Dual basis is "parent functionals" that generate all others

So the dual basis is exactly like a "parent function" set — it's the *coordinate system for functionals*.

1. V: The Primal Space

- V contains the **vectors themselves** — positions, states, forces, whatever your problem is modeling.
- It's the "raw material" of the system: the things that can be combined and transformed.

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2. V^*: The Dual Space

- V^* contains the **linear measurements** — the ways you can probe, observe, or extract information from V.
- Each $\varphi \in V^*$ is a "question" you can ask of a vector: *"What is your component in this direction?"*
- Together, all functionals describe every possible linear measurement.

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3. Together: The Whole Perspective

- V tells you **what the system *is*** (its possible states).
- V^* tells you **how the system can be *observed or tested*** (all possible linear probes).

The interaction $\varphi(v)$ (a scalar) is where **state meets measurement** — the result of observing v through the "lens" of φ.

<u>Matrix Representation of a Linear Map</u>

Matrices are used to represent linear maps. They are presented as arrays of numbers; arranged in rows and columns.

A matrix $A \in \mathbb{F}^{m \times n}$ represents a linear map

$$T : \mathbb{F}^n \longrightarrow \mathbb{F}^m, \qquad T(x) = Ax.$$

- **Domain:** $\mathbb{F}^n$ (input space, dimension n)
- **Codomain:** $\mathbb{F}^m$ (output space, dimension m)

The dimension of the matrix is according to the dimensions of the domain and the codomain.

A square matrix will have domain and codomain with same dimensions. If a matrix is square and invertible, the linear transformation is an isomorphism.

Non-square matrices have the linear map involving a change in dimension. While linearity, addition, and scaling are preserved, there is loss of one-one correspondence. Linear maps in these cases are simple vector space homomorphisms.

A matrix is in m x n if the LT involves moving from a n dimensional vector space to a m dimensional vector space.

A 2 x 3 matrix will have 2 rows and 3 columns. Through matrix multiplication, it will act on a domain vector, a 3 x 1 vector to yield a 2 x 1 vector, the image of the vector under the transformation.

Column and Column Space

A matrix can be split vertically in terms of columns. Each column of the matrix represents the image of one of the domain's bases under the linear map. Naturally, the number of columns matches the number of bases in the domain*.

*To ensure all bases get transferred, the columns must be linearly independent to ensure each independent bases of the domain gets mapped.

Suppose we have a matrix, $A = \begin{matrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{matrix}$

If we were moving from T: R^3 to R^2 and assume standard bases,

The transformation would be as follows:

$$\begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \; --> \begin{matrix} 1 \\ 3 \end{matrix}$$

$$\begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \; --> \begin{matrix} 2 \\ 0 \end{matrix}$$

$$\begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \; --> \begin{matrix} 0 \\ 1 \end{matrix}$$

As can be seen, for any Ax, the matrix A will take a 3D vector and output it in 2D space. These new coordinates may or may not serve as the new bases for the image.

These one is linearly dependent; they cannot be used as the new bases. We lose the grid of reference we had in the domain and need to find a new set of bases for the image.

However, something interesting is to be noted, take any other vector than the bases in the domain. The vector is generated by the bases.

e.g. $\begin{matrix} 3 \\ 4 \\ 5 \end{matrix} = 3 \,.\begin{matrix} 1 \\ 0 \\ 0 \end{matrix} + 4 \,.\begin{matrix} 0 \\ 1 \\ 0 \end{matrix} + 5 .\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}$

The columns show us how the generator changes. Suppose we want to know where $\begin{matrix} 3 \\ 4 \\ 5 \end{matrix}$ will land, we can scale each component of the vector in the domain by the transformation of the generators. If the generators make the domain vectors and we know the change in the generators, the change can be carried to the domain vectors to make the image.

This amounts to scaling each component of the input vector by the bases/generator image

$$T(\begin{matrix} 3 \\ 4 \\ 5 \end{matrix}) = 3 \,.\begin{matrix} 1 \\ 3 \end{matrix} \; + \; 4 \,.\begin{matrix} 2 \\ 0 \end{matrix} \; + \; 5 \,.\begin{matrix} 0 \\ 1 \end{matrix} \; = \; \begin{matrix} 11 \\ 14 \end{matrix}$$

This is one vector in the image. The transformation can sometimes allow us to reach the whole codomain or sometimes only a subset of the codomain, known as the image of the transformation. It can be known by collecting all possible linear combinations of the columns, not just for that unique vector above, but all possible input vectors. This image of the transformation is also known as the column space of the matrix.

Rows and Row Space

The matrix, we can see as a collection of row vectors. Suppose we have a matrix,

$$A = \begin{matrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{matrix}$$

Each row can be seen as a co-vector; a linear functional. It is one which produces a scalar taking a vector as input. There are 2 co-vectors for that transformation, each corresponding to a component of the image, through matrix multiplication and dot product, we obtain the scalars for each component of the output vector

$$\begin{pmatrix} 1 & 2 & 0 \end{pmatrix} . \begin{matrix} 3 \\ (4) \\ 5 \end{matrix} = 11$$

$$\begin{pmatrix} 3 & 0 & 1 \end{pmatrix} . \begin{matrix} 3 \\ (4) \\ 5 \end{matrix} = 14$$

$$\begin{pmatrix} 11 \\ 14 \end{pmatrix}$$

The collection of the co-vectors of the transformation forms the row space of the corresponding matrix. It is a collection of all linear functionals. They form a vector space on their own and live in a subset of the dual of the domain. The row space being a linear combination of the rows represent all the possible ways that the linear functionals of the matrix can probe/measure input vectors.

Note: The scalar that the covectors output are not the covectors, the covectors are the measuring device of the linear transformation.

Parts of a Whole

Column Picture

You get where the bases are going, from that, you reconstruct your desired vector from the rules of transformation of the bases. The space is morphing and carrying everything with it

I know that my path is bad. From my past experience, I always fell back onto what the elders told me.

I can see, I can observe society and know that this trend is bad, those similar to me fell into a place I want to avoid, but something is pulling me.

I am not satisfied with merely constructing a future I want, I know I should just do this and do that, pass my exams, go to uni, get my masters, get a job, get a family, something is pulling me somewhere else, I know for fact, I am not losing myself into dreams like Don Quixote, whatever path I go on, I will still end back again into society.

The column picture shows me the advice of wisdom onto how to construct a life I want, I just need to have faith into the process, no need to understand, most of what I understood, I discarded at the end of the journey in favour of wisdom.

I can't have faith, I am so jealous of those who do, I need to understand.

Why understand, because I fear something in me, some part of me will lead to ruin if I follow the path. I can't just jump into the transformation seeing all the ideals/the statistics showing where the basis gets mapped and assume that I will land in part of that space. What if I die and get ruined if I take a leap of faith? What if I am missing something me, and my efforts are meaningless?

The column picture lacks what gets mapped to the kernel (or constant affine map)
Where do I go?

<u>Row Picture</u>

I need to know. Whatever measurement I can get, whatever library of knowledge opens itself to me; whatever measuring device I can invent. Component by component I must understand myself. Fitness: Strength, Speed, Stamina, Suppleness, Skill, Intelligence, Fluid, Crystal; Dating, height, income, jawline, muscles.

I must measure all, component by component, I know myself, what is my greatest strength, my greatest weakness, what is futile for the transformation, what will go to the kernel; being kind never mattered; it's all money and confidence. It is hard being honest with oneself.

What is harder is yet to come; I know myself piece by piece, but I cannot see myself, I do not know how I fit into this whole new vector space, where is my place? I cannot see where I stand. I know the measurement I am piece by piece but not how I assemble into the final transformation.
I seek validation from people. I need to know where I stand. My head is too filled I know too much, I need to depend on others to cut away all my confusing thoughts; it is torture.

<u>Pivot Upside Down</u>

I just want to know back what's important and what to ignore; so much was illusions: which part of me matters to the transformation I seek. Pivots show which input directions matter, what actually was me losing time and not contributing to my desire; lost myself on the way; and which output directions are truly independent and worth walking. Pivots link row measurement to an output direction; linking kernel information (row) to image structure (column). All the independent pieces of information are captured by the pivots; everything else (non-pivot rows/columns) is redundant or dependent.

I see within and without

Hyperplanes, Rows and Co-vectors

A matrix is the coordinate representation of a linear transformation.

Each row of a matrix is a co-vector measurement involved in the transformation. The i^{th} row is measures for the i^{th} coordinate in the output and is made up of all the i^{th} coordinates in the image bases.

The number of rows equals the number of bases of in the image.

Each column corresponds to the image of the bases under the transformation.

[Note: Any vector space of dimension n will have n bases and thus n components for each basis vector]

Remember that in the domain, the input query is a linear combination of the bases as it is an element of the domain.

The co-vector/linear functional of the first-row measures how much the shift from the one domain basis to the new one has an effect on the first coordinate of an output, then it measures how much the second to new second, third to new third has an effect on the first coordinate in the image. It then uses linear combinations in domain to do the weighted sum that each component change has on the first component.

Each component isolated narrows down the dimension of the hyperplane until a single point is reached. This is the image of the vector under T.

Return to Abstract Vector Spaces

Matrix theory encodes how a linear map behaves, expressed in coordinates.
A represents T : V → W

V is a vector space containing vectors. W is also a vector space. Both bear their respective dimensions and bases.

V* is the dual space of V. It is the vector space that contains all the co-vectors/linear functionals on V. W* is the same.

They each have their dual basis, whereby all linear functionals are a combination of these.
For finite-dimensional spaces, V and V* have the same dimension.

The image of the transformation, Im(T) is a subset of the codomain W.
In coordinates term, the span of the image of the bases forms the column space.
Like Im (T), Col (A) is a subset of the codomain W

The row space is a subset of the dual of the domain, V*. It contains all the linear functionals involved in making a measurement for the transformation.
The row space tells us which linear condition are being checked when we apply T.

Kernels

Algebraic structures like vector spaces have a kernel or a null point. Like a singularity.

A vector space has only 0 as the null point. For any other non-zero constant maps, we require affine spaces.

In coordinates term, singular matrices collapse one dimension from the domain. That is the kernel of the transformation is not only 0 maps to 0; non-zero vectors also get sent to 0. The kernel is non-trivial.

Ker(T) = {all v of Dom | T(v) = 0}
It is all the inputs of the domain that get sent to 0 under the transformation.

Rank Nullity Theorem
The rank of a matrix is the dimension of the column or row space; these two are equal. It is all the linear functionals involved in the matrix and all the bases where the image is projected. These two are equal for a linear transformation.
The rest of the part of the domain not seen by the transformation gets sent to the kernel.
Hence, dim (Dom) = dim (Im(T)) + dim (ker(T))

0-vector
The 0 vector of the co-domain is always in the image of the transformation. Linearity conditions force 0 to map to 0; 0 of Dom always map to 0 of Cdm.
Also, 0 is the point fixed by every LT; it is the unique anchor point that cannot move under any linear map.

Remember: For a vector to be sent to the kernel, Ax = 0
Ax is equal to the dot product between each row of A and x.

$Ax = 0$ means that x is orthogonal to all rows, hence the row space
This implies that elements of the kernel, the null space is <u>orthogonal</u> to the row space

The column space has no such relationship with the null space as the latter lives in Dom while the column space is in the Cdm.

However, if we transpose A such that the rows of A are the columns of A^T
$Row(A^T) = Col(A)$
From the previous argument, this implies that null (A^T) is orthogonal to $Row(A^T)$
From $Row(A^T) = Col(A)$
Col (A) is orthogonal to the null space of the transpose of A

- **Null(A):** inputs invisible to all row measurements.
- **Row(A):** the measurements themselves. Null and row space are perpendicular.
- **Col(A):** all possible outputs.
- **Null(A^T):** output directions that no input can reach. Those are orthogonal to the actual column space.

Dual Map T*

Every linear map T : V --> W, from one vector space to another induces a dual map/transpose map T* : W* --> V*.

Let V and W be vector spaces over the same field F

We have a linear map T : V --> W

The dual spaces are:

V* = {linear functionals f : V→ F}
W* = {linear functionals g : W→ F}

T maps from V to W
g maps from W to F

By function composition, g ∘ T : V → F,

This composition essentially sends a vector from V to its field, hence is a linear functional in V*.

We can denote that composition by T*(g) : = g ∘ T valid for any g in W*

T*(g) lives in V*

Therefore, every linear map has a corresponding dual map, which in coordinate form is transpose of the matrix A.

The row space is the image of the dual map T*: W* --> V*

It is the set of co-vectors on V that arise by pulling back co-vectors from W

<u>Transpose</u>

V, W are vector spaces with vectors as elements

V^*, W^* are the corresponding dual spaces containing co-vectors

We have the linear map $T : V \to W$

If T is an isomorphism, we have $T^{-1} : W \to V$

All linear maps T induces a dual map between the dual spaces

$T^* : W^* \to V^*$

This is known as the transpose map.

<u>This map is a natural consequence of T</u>

All vector spaces have dual spaces with functionals sending vectors to underlying field

T sends from V to W.

Functional g sends from W to F

$g \circ T$ will send from V to F,

fulfilling the condition for being in V^*

We can then make a function taking g in W^* and composing with T

$T^*(g) := g \circ T$

Therefore $T^* : W^* \to V^*$

for all $T : V \to W$

The induced transpose map only occurs in direction $W^* \to V^*$

We cannot have $V^* \to W^*$, because linear functionals are defined on their respective vector spaces, we need to send W back to V, requiring inverses or inner product structure via representation theorem representing a co-vector as a vector.

Riesz Representation Theorem

In simple words, it says that in an inner-product space, a co-vector can be represented by a vector.

In a general vector space, a vector is an element of V. a co-vector is a linear functional, element of V*. There is no way to identify one with the other.

For every vector v in V (inner product)
We can associate a co-vector $f_v(w) = v . w$
Noting f_v is an element of V*
Through the dot product, we used a vector as a linear functional, this can be done for any vector.

Like the vector space is based on linear combinations, so is the dual space; this principle spans the entire of both spaces, setting up an isomorphism between V and V*.

In these spaces, co-vectors can be interpreted as vectors, this is why in R^n (inner product) in the coordinates representations the transpose is like making the rows of a matrix become the columns and the columns the rows.

We cannot switch vectors and co-vectors, they are different. But through the representation theorem in an inner product space, this can be done.

In **general vector spaces**: transpose acts between duals ($W^* \to V_*$)

In **Hilbert/Euclidean spaces**: the inner product lets us identify vectors with covectors, so the transpose can be seen as acting directly between vectors, and that's why we can swap rows and columns in coordinates.

3. Why the transpose works cleanly

Now suppose $A : V \to W$.
The transpose is defined as:

$$\langle Av, w \rangle_W = \langle v, A^T w \rangle_V.$$

This definition only makes sense because the dot product allows us to:

- Write $w \in W$ both as a vector and as a covector (via $f_w(x) = w \cdot x$).
- Write $v \in V$ both as a vector and as a covector.

So the transpose moves rows ↔ columns **without messing up the distinction** because the dot product *quietly identifies* those two spaces.

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4. Without the dot product

If we had no dot product, V and V^* would remain separate, and we couldn't just flip rows ↔ columns.
In that setting:

- A matrix would represent a map $V \to W$.
- The "transpose" would represent a different map $W^* \to V^*$.
  But you couldn't compare those objects directly.

It's only because the dot product equates vectors with covectors that we can reuse the same numerical array and interpret "rows ↔ columns" as valid.

<u>Bilinear Maps</u>

It is a function combining elements of 2 vector spaces to yield element of a third space.

$B : V \times W \to X$ where V, W, X, are vector spaces over same F

$\forall\, w \in W$, where the map B_w denotes fixing w.

$B_w : v \mapsto B(v, w)$ is a linear map from V to X

The same can be said for:

$B_v : w \mapsto B(v, w)$ being a linear map from W to X

Such a map B satisfies the following properties.

- For any $\lambda \in F$, $B(\lambda v, w) = B(v, \lambda w) = \lambda B(v, w)$.
- The map B is additive in both components: if $v_1, v_2 \in V$ and $w_1, w_2 \in W$, then $B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$ and $B(v, w_1 + w_2) = B(v, w_1) + B(v, w_2)$.

Matrix multiplication is a bilinear map $M^{n \times p} \times M^{p \times m} \to M^{n \times m}$

The cross-product is also a bilinear map

<u>Bilinear forms</u>

It is a special bilinear map where the arguments are from the same vector space and the co-domain is the base field F

$B : V \times V \to F$

Example of bilinear form in vector spaces with inner product is the inner product.

Bilinear form Matrix

Choosing a basis $\{e_1, \ldots, e_n\}$ for V, the bilinear form is determined on how it acts on the bases. The bilinear form B (x, y) is given by a n x n matrix A.

$A_{ij} = B\ (e_i\ , e_j)$

The bilinear form evaluates to: x^TAy

In R^n the Inner product $<x\ , y>$ is a special bilinear form x^TIy with I being the matrix

Step 1. Define a bilinear form

Let's pick:

$$B\big((x_1, x_2), (y_1, y_2)\big) = 2x_1y_1 + 3x_1y_2 + 4x_2y_1 + 5x_2y_2.$$

So it takes two vectors in $\mathbb{R}^2$ and returns a number.

Step 2. Build the matrix

Use the basis $e_1 = (1, 0)$, $e_2 = (0, 1)$.

- $A_{11} = B(e_1, e_1) = 2$
- $A_{12} = B(e_1, e_2) = 3$
- $A_{21} = B(e_2, e_1) = 4$
- $A_{22} = B(e_2, e_2) = 5$

So the matrix is

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

Theorem: Every Bilinear from B on V defines a pair of linear maps from V to its dual, V*.

$B_{<\cdot\,,\,\cdot>} : V \times V \to F$, we are taking 2 elements $x, y \in V$ and outputting a scalar

Fixing one input each time allows us to produce 2 functions

Fixing a vector x, we can define a co-vector

$f_x : V \to F$,

$f_x : y \mapsto B\ (x, y)$

Similarly, fixing y,

$g_y : V \to F$,

$g_y : x \mapsto B\ (x, y)$

Both f and g are linear functionals on V,

So, $f_y, g_x \in V^*$

We can define a function $\phi\ (x) = f_x$ which fulfils $V \to V^*$ and is a valid linear function on V

Same for $\phi\ (y) = g_y$

Φ assigns to each vector y a co-vector $\phi\ (y)$

We showed such a linear function $V \to V^*$ can exist

Show Linearity

3. Check linearity of Φ

Take $y_1, y_2 \in V$ and $\alpha, \beta \in \mathbb{F}$:

$$\Phi(\alpha y_1 + \beta y_2) = f_{\alpha y_1 + \beta y_2} = (x \mapsto B(x, \alpha y_1 + \beta y_2)).$$

Because B is linear in the **second argument**, we have:

$$B(x, \alpha y_1 + \beta y_2) = \alpha B(x, y_1) + \beta B(x, y_2) = \alpha f_{y_1}(x) + \beta f_{y_2}(x).$$

This holds for **all** x, so

$$\Phi(\alpha y_1 + \beta y_2) = \alpha \Phi(y_1) + \beta \Phi(y_2).$$

✅ That proves $\Phi : V \to V^*$ is **linear**.

4. In matrix terms

Let's say $V = \mathbb{F}^n$ and the bilinear form has matrix A, so

$$B(x, y) = x^\top A y.$$

- Fix y. Then

  $$f_y(x) = x^\top A y = (Ay)^\top x.$$

  That is, f_y corresponds to the covector $(Ay)^\top \in V^*$.
  So the map $y \mapsto f_y$ is exactly the linear map $y \mapsto Ay$ (then viewed as a covector).
- Fix x. Then

  $$g_x(y) = x^\top A y = (A^\top x)^\top y.$$

  That is, g_x corresponds to the covector $(A^\top x)^\top$.
  So the map $x \mapsto g_x$ is the linear map $x \mapsto A^\top x$.

So the two associated maps are represented by A and $A^\top$.

Inner Product

When B is symmetric, the above maps co-incide, $A^T = A$

1. Start with a bilinear form $B : V \times V \to \mathbb{F}$

We have two canonical maps from $V \to V^*$:

1. **Fix the first slot:**

$$\Phi : V \to V^*, \quad \Phi(x) = B(x, -).$$

2. **Fix the second slot:**

$$\Psi : V \to V^*, \quad \Psi(y) = B(-, y).$$

So every $x \in V$ gives a functional $\Phi(x)$ acting on vectors $v \mapsto B(x, v)$, and similarly for $\Psi(y)$.

3. Symmetric bilinear forms

If B is **symmetric**, i.e.

$$B(x, y) = B(y, x),$$

then for every $x \in V$:

$$\Phi(x) = B(x, -) = B(-, x) = \Psi(x) \in V^*.$$

✅ So the two maps Φ and Ψ **coincide**.

- That is, there is now **one canonical map** $V \to V^*$ induced by B.
- If B is also **nondegenerate**, this map is an isomorphism: $V \cong V^*$.

1. Definition (for bilinear forms)

A bilinear form $B : V \times V \to \mathbb{F}$ is **nondegenerate** if:

$$\text{If } B(v, w) = 0 \text{ for all } w \in V, \text{ then } v = 0.$$

Equivalently:

$$\text{If } B(v, -) = 0 \text{ as a functional in } V^*, \text{ then } v = 0.$$

- This says that the map $\Phi : V \to V^*,\ v \mapsto B(v, -)$ is **injective**.

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2. Intuition

- Nondegeneracy means **no nonzero vector is "invisible"** to the bilinear form.
- Every nonzero $v \in V$ interacts with **some** vector w to produce a nonzero scalar.

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3. Matrix criterion

If V is finite-dimensional and B has matrix A in some basis:

- B is nondegenerate **iff** A is invertible (i.e., $\det A \neq 0$).

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4. Why it matters

- If B is symmetric and nondegenerate, the map $\Phi : V \to V^*$ is an **isomorphism**.
- That's the linear-algebraic core of identifying $V \cong V^*$ via a bilinear form.
- If B were degenerate, some vectors would map to $0 \in V^*$, so the map would **not be invertible**.

If B is positive definite, non-degenerate, and symmetric, the bilinear form is an inner product. This is why in inner product space; the isomorphism allows a co-vector to be a vector through isomorphism. This is the Riesz Representation Theorem for finite vector spaces.

Canonical Mapping and Canonical Evaluation to V**

what if the map is combing an element of V x V* to F?

Nice — that's a different but very fundamental situation.

If the map is

$$\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{F},$$

then you are talking about the **canonical pairing** (or evaluation map).

Definition

For $v \in V$ and $\phi \in V^*$ (a linear functional), define

$$\langle v, \phi \rangle := \phi(v).$$

So you plug the vector v into the functional ϕ, and get a scalar in $\mathbb{F}$.

Properties

- It's **bilinear**:
  - Linear in v: $\phi(v1 + v2) = \phi(v1) + \phi(v2)$.
  - Linear in ϕ: $(\phi1 + \phi2)(v) = \phi1(v) + \phi2(v)$.

So this is exactly a bilinear map from $V \times V^*$ to $\mathbb{F}$.

Matrix picture

If $V = \mathbb{F}^n$ with standard basis,

- a vector is a column $v \in \mathbb{F}^n$,
- a covector (element of V^*) is a row $\phi \in (\mathbb{F}^n)^* \cong \mathbb{F}^{1\times n}$.

Then the pairing is just matrix multiplication:

$$\phi(v) = \phi\, v \quad (\text{row} \times \text{column} = \text{scalar}).$$

✅ So:

- **Bilinear forms**: $V \times V \to \mathbb{F}$.
- **Canonical pairing**: $V \times V^* \to \mathbb{F}$.

Both are bilinear maps, but the second one uses one slot for a vector and one for a covector.

Do you want me to explain how this pairing gives an **isomorphism** $V \cong V^{**}$ (the double dual)?

1. Setup

We have a vector space V over a field $\mathbb{F}$.

- Its **dual space** V^* is the space of linear functionals $V \to \mathbb{F}$.
- Its **double dual** V^{**} is the dual of V^*, i.e. linear functionals $V^* \to \mathbb{F}$.

So elements of V^{**} are functions that eat a covector and return a scalar.

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2. Canonical pairing

Recall the canonical pairing:

$$\langle v, \phi \rangle := \phi(v), \quad v \in V,\ \phi \in V^*.$$

This is bilinear.

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3. From pairing to isomorphism

We can use this pairing to turn any $v \in V$ into an element of V^{**}.

Define a map

$$\iota : V \to V^{**}$$

by

$$\iota(v)(\phi) = \phi(v).$$

- Input: a vector v.
- Output: a functional on V^* (i.e. an element of V^{**}).
- That functional acts on $\phi \in V^*$ by evaluation.

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4. Why is this an isomorphism?

- **Linearity**:
  $\iota(v1 + v2)(\phi) = \phi(v1 + v2) = \phi(v1) + \phi(v2) = \iota(v1)(\phi) + \iota(v2)(\phi)$.
  Same for scalar multiples.
- **Injectivity**:
  If $\iota(v) = 0$, then $\phi(v) = 0$ for all ϕ. The only vector annihilated by *all* functionals is $v = 0$.
- **Surjectivity** (for finite-dimensional V):
  $\dim(V) = \dim(V^*) = \dim(V^{**})$. Since ι is an injective linear map between two spaces of the same finite dimension, it is also surjective.

So, in finite dimensions:

$$V \cong V^{**}.$$

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5. Infinite-dimensional note

If V is infinite-dimensional, ι is still injective but not always surjective. So we only get a natural embedding, not a full isomorphism.

✅ So the canonical pairing $V \times V^* \to \mathbb{F}$ is the ! to constructing the isomorphism between V and V^{**}.

Perfect — let's pin this down in the **matrix picture**.

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1. Vectors and covectors

Take $V = \mathbb{F}^n$.

- A vector $v \in V$ is a **column**:

$$v = \begin{bmatrix} v1 \\ \vdots \\ vn \end{bmatrix}.$$

- A covector $\phi \in V^*$ is a **row**:

$$\phi = \begin{bmatrix} \phi 1 & \cdots & \phi n \end{bmatrix}.$$

Then the pairing is just matrix multiplication:

$$\phi(v) = \phi\, v = \sum_{i=1}^{n} \phi i v i \in \mathbb{F}.$$

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2. Double dual V^{**}

Elements of V^{**} are linear maps $V^* \to \mathbb{F}$.
So, given a $v \in V$, we can define a functional $\iota(v)$ on V^*:

$$\iota(v)(\phi) = \phi(v).$$

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3. In coordinates

Fix the standard basis $e1, \ldots, en$ of V, with dual basis $e\overset{*}{1}, \ldots, e\overset{*}{n}$ in V^*.

- If $v = \sum vi ei$,
- then for each dual basis element,

$$\iota(v)(e\overset{*}{j}) = e\overset{*}{j}(v) = vj.$$

So the action of $\iota(v)$ on the dual basis **recovers the coordinates** of v.
This shows ι is literally "copying" the column vector into the double dual.

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4. Relation to transpose

Here's the key:

- If $A : V \to W$ is a linear map with matrix A,
- its dual map $A^* : W^* \to V^*$ has matrix $A^\top$.

Now, under the identification $V \cong V^{**}$,

- the double dual map $A^{**} : V^{**} \to W^{**}$ corresponds exactly to A.
- That is: the transpose action cancels out when you go to the double dual, and you get back the original matrix.

So the transpose is how covectors transform, but the double transpose gives back the original transformation on vectors.

5. Summary in matrix terms

- Vector v: column.
- Covector ϕ: row.
- Pairing: row $\times$ column = scalar.
- Double dual map: send column to functional that extracts coordinates → isomorphism $V \cong V^{**}$.
- Under this isomorphism, linear maps and their transpose-duals fit perfectly: $A^{**} = A$.

what is the annihilator in vector spaces?

The **annihilator** is a construction that links a subspace of a vector space to a subspace of its dual.

Let's break it down:

- Suppose V is a vector space over a field F.
- Its dual space V^* is the set of all linear functionals $f : V \to F$.
- If $W \subseteq V$ is a subspace, then the **annihilator of W**, denoted W°, is defined as

$$W^\circ = \{f \in V^* \mid f(w) = 0 \text{ for all } w \in W\}.$$

So:

- It's the set of all covectors (linear functionals) that "kill" every vector in W.
- Geometrically, you can think of it as the collection of linear measurements that see no component in W.

Examples:

1. **Trivial cases:**
   - If $W = \{0\}$, then $W^\circ = V^*$, since every functional vanishes on 0.
   - If $W = V$, then $W^\circ = \{0\}$, since the only functional that vanishes on all of V is the zero functional.
2. **In $\mathbb{R}^3$:**
   Take $V = \mathbb{R}^3$ with dual $V^* \cong \mathbb{R}^3$ (via the dot product).
   If W is the span of one vector, say $W = \mathrm{span}\{(1, 0, 0)\}$, then

$$W^\circ = \{f \in V^* : f(1, 0, 0) = 0\}.$$

   In the dot product identification, this corresponds to all vectors perpendicular to $(1, 0, 0)$, i.e. the plane $\{(0, y, z)\}$.

Setup

Let $T : V \to W$ be a linear map, with matrix A (say $m \times n$) in some bases:

$$A : \mathbb{F}^n \to \mathbb{F}^m, \quad Ax = y.$$

- **Kernel:**

$$\ker(A) = \{x \in \mathbb{F}^n : Ax = 0\}.$$

- **Row space and annihilator:**
  The rows of A are covectors (functionals on $\mathbb{F}^n$).
  Each row r_i acts on x by $r_i \cdot x$.
  The set of all rows spans a subspace of $(\mathbb{F}^n)^*$, called the **row space**.

Now notice:

$$\ker(A) = \{x \in \mathbb{F}^n : r(x) = 0 \text{ for all rows } r\}.$$

That is, the kernel consists of exactly those vectors annihilated by all the row functionals.
So the **kernel = the annihilator of the row space**.

Flip it with the transpose (dual map)

The dual map is $A^T : (\mathbb{F}^m)^* \to (\mathbb{F}^n)^*$.

- Its kernel is

$$\ker(A^T) = \{f \in (\mathbb{F}^m)^* : f(Ax) = 0 \ \forall x\}.$$

  Which is precisely the annihilator of the **column space of A**.

Summary in matrix terms

- $\ker(A)$ = all vectors killed by all rows (annihilator of row space).
- $\ker(A^T)$ = all covectors killed by all columns (annihilator of column space).

So kernels and annihilators are two sides of the same coin, depending on whether you look at the map or its transpose/dual.