# Probability

- The sample space $\Omega$ is the set of all possible outcomes.
  e.g. $\Omega = \{H, T\}$, $\{1,2,3,4,5,6\}$, $\{R, B, G, Y, O, P\}$
  ↘ single ~~coin fli~~ coin flip ↘ Single dice toss
- *An event specifies the outcome we are interested in. It is a subset of the sample space.
  e.g. $\{4\}$ rolling 4
  $\{2,4,6\}$ rolling an even number
  The elements of the event space all satisfy the event specified and none other.
- A $\sigma$-algebra ($\mathcal{F}$) is a collection of subsets of the sample space $\Omega$ that satisfy the following properties.
  $\Omega \in \mathcal{F}$
  $\emptyset \in \mathcal{F}$
  if $A \in \mathcal{F}$ then $\Omega \setminus A \in \mathcal{F}$
  if $A_1, A_2, A_3, \ldots \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

  You don't need to think to live.
  You don't need to be smart to live.
  You don't need to solve problems to live.
  Don't think too much.

  These axioms allow for a measure space $(\Omega, \mathcal{F}, \mu)$ to be defined. Those subsets which are measurable ~~can be~~ are in the $\sigma$-algebra and can be assigned probabilities

  * An event is a subset that: 1) Satisfy the event specified
  2) belong to $\mathcal{F}$.

  This ensure that probability can be defined for these events.

  → Events are elements of the $\sigma$-Algebra and the latter determines which subset of $\Omega$ are valid events.

- A probability space $(\Omega, \mathcal{F}, P)$ is a measure that ensures all elements of $\mathcal{F}$ sum up to 1.
  It obeys the axioms of the $\sigma$-algebra as well as the following:
  $P(\emptyset) = 0$
  $P(\Omega) = 1$
  $P(A) \geq 0$
  $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

These ensure that probability is calculated as traditionally done.

Collection of stuff

Bring it into being.

## Philosophy

Phenomenology.
↳ Measurements is inherently observer dependant. Measurement is inseparable from some choice or selection

X

Piece of land

Some Set X being a collection of stuff

Landscape profile or topology map

$(X, \mathcal{J})$

$\mathcal{J}$ is a collection of subsets of X such that smooth "connections" between data points is meaningful.

A topology on X provides the "Shape" ~~and~~ of the data being the "closeness" of points in the set.

Can you measure something w/o choosing to measure it?

Question, why do ? specifically define a $\sigma$-algebra.

$(X, \sigma)$

$\sigma$ - Algebra (measurable space)

What do we want to measure is a good question.

elevation Height of each region of the map or size of rock formation

*

This is done by the measure function $\mu$.

Implicitly choosing the parts we will measure.

Partitioning land into parts which can be measured.

Size of rocks
Height of land
Dryness of land

A $\sigma$ - algebra (allows us to measure the "volume" / "importance" of each of these shapes on the set.)

Split the land in such a way that it can be measured.

How to measure

↳ $\boxed{\text{What even is measuring stuff ?}}$

↳ Philosophy

$\mu$ : Measure Space

$(X, \sigma, \mu)$

↳ Info about the size of each land.

· Axioms : $\emptyset$ & $\Omega$ included

or $\infty$

0 100

Smallest & largest element included.

$\Omega$

$\{ \emptyset \mid \}$

How do I interpret complement & union in terms of measuring land height

Height of villages above sea level → $\emptyset$

+

Measure & Measurable Space = Measure Space

$\mu$ $(X, \Sigma)$ $(X, \Sigma, \mu)$

$\{ \emptyset, \Omega$; Countable union closure; Complement Closure $\}$

$\geq 0$

Countable additive

Elements of a measurable Space are measurable Subsets

Why do the axioms make them measurable?

What can we measure?

* A measurable Space is a set equipped with a $\sigma$-algebra Structure.

$(X, \Sigma)$

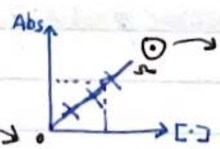
→ Collection of subsets of $X$.

You can't measure something without choosing to measure it.

In our case, a Subset would correspond to choosing a region of the landscape profile.

Axioms : There is an empty region ($\emptyset$) and the whole region ($X$ or $\Omega$) as part of the measurement Structure.

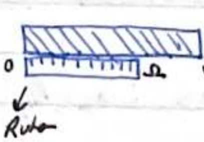
① The measurement System is closed.




Same as choosing even.

② Closed under complementarity.

↳ When we choose a subset $A$ of landscape, we implicitly state that it is not $A^c$ (relative to universal set $\Omega$)





③ Closed under union

↳ Choosing piece of land A & B together makes it as a single choice of larger piece of land C

$C = A + B$

# Pushforward Measure

A pushforward measure is obtained by transferring a measure from one measurable space to another using a measurable function.

Given measurable spaces $(X_1, \Sigma_1)$ and $(X_2, \Sigma_2)$, a measurable mapping $f: X_1 \to X_2$ and a measure

$\mu : \Sigma_1 \to [0, +\infty]$,

the pushforward of $\mu$ is defined to be the measure $f_*(\mu): \Sigma_2 \to [0, +\infty]$ given by

$$\boxed{f_*(\mu)(B) = \mu(f^{-1}(B)) \text{ for } B \in \Sigma_2}$$

The pushforward measure is also denoted $f_{\#}\mu$

$\mathcal{F}$ ensures measure of events occur properly

this is defined as $\mathbb{P}$ on $\mathcal{F}$

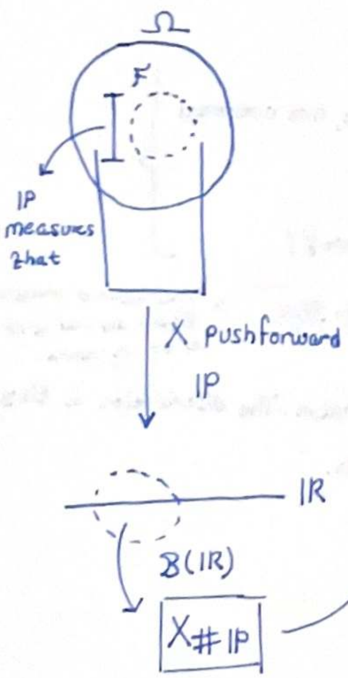
In terms of probability, A random variable pushforward the probability measure $\mathbb{P}$ defined on a probability space to the real line (equipped with a Borel $\sigma$-algebra).

To also allow for measure to be defined consistently.

$$\boxed{\underbrace{\mathbb{P}_X(B)}_{X_{\#}\mathbb{P}} = \mathbb{P}(X^{-1}(B)) \text{ for } B \in \mathcal{B}(\mathbb{R})}$$

$\mathbb{P}_X$ reflects the distribution of the probability on $\mathbb{R}$.

This was evaluated by $\mathbb{P}$ involving $\mathcal{F}$ & $\Omega$




You need a good number system to be able to write probability density functions.
The pushforward transfers that probability to that number system so it can be captured in equations and interpreted numerically.

This is the thing that is captured by Probability density functions.

$\mathbb{P} : \mathcal{F} \to [0, 1]$

Now, the measure $\mathbb{P}$ must also in some way involve $\Omega$ for the ~~measure~~ notion of Probability to be relevant.