The Product of Shapes

The product is the result of a multiplication operation. The latter is present in all areas of mathematics.

Generally, multiplication takes 2 elements as input and produces another one based on a set of pre-defined rules.

There are different kinds of product which are all defined differently based on their utility and consistency with mathematics.

$2 \times 3 = 6$ $\quad -2 \times -2 = 4$ $\quad \pi \times \frac{1}{2} = \frac{\pi}{2}$ $\quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = (a \cdot d) + (b \cdot e) + (c \cdot f)$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} ae+bf \\ ce+df \end{pmatrix}$

$A \times B = \{(a,b) \mid a \in A, b \in B\}$

The product is usually introduced during the multiplication of Natural numbers.

The arithmetic for Natural numbers is formalised in the Peano Axioms.

Only a short segment will be presented here to keep this article light and relevant. More details can be found in the provided references.

- We define a function $S : \mathbb{N} \rightarrow \mathbb{N}$ such that S maps an element of $\mathbb{N}$ to its successor.

[Intuitively, we think $S : x \mapsto x+1$ but we have not yet defined addition and will not do so here.]

- We define a least element in $\mathbb{N}$ (It is the successor of none) and assign it the symbol 0.

e.g. $S(0) = 1$ ← (Symbol for successor of least element 0)

$S(S(0)) = S(1) = 2$ (Symbol for successor of 1)

and so on...

- We now define multiplication based on the following rule: $\overset{\text{I}}{a \cdot 0 = 0}$, $\overset{\text{II}}{a \cdot S(b) = a + (a \cdot b)}$

where $a, b \in \mathbb{N}$, $+$ is the addition operator [behaves as usual] and $\cdot$ is the multiplication operator.

Let's try $7 \cdot 3 = 21$

$7 \cdot S(2) = 7 + (7 \cdot 2)$ Rule II and $S(2) = 3$

$= 7 + (7 \cdot S(1))$ $S(1) = 2$

$= 7 + (7 + (7 \cdot 1))$ Rule II

$= 7 + (7 + (7 \cdot S(0))$ $S(0) = 1$

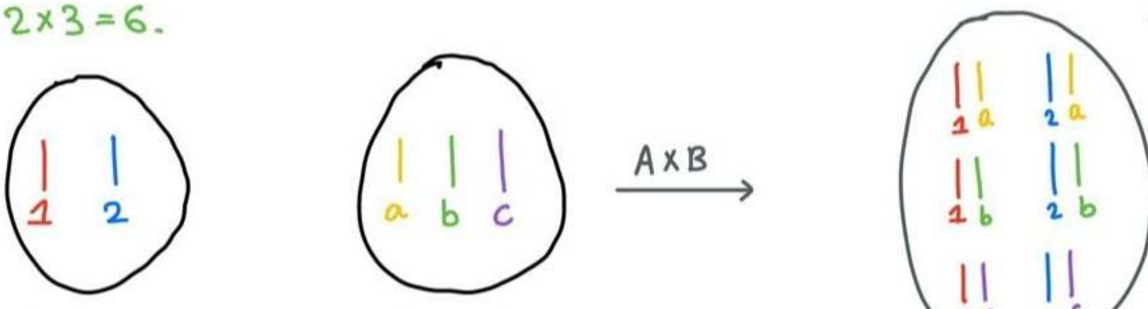
$= 7 + (7 + (7 + (\cancel{7 \cdot 0})))$ Rule I

$= 7 + 7 + 7 = 21$

We see here how this definition of multiplication 'unpacks' a multiplication into repeated addition!

As we have seen, the Peano Axioms formalises the usual and historical definition of multiplication as repeated addition. In the past, this sped up counting of cattle, sack of grains in agriculture and collection of taxes and debt.

Apart from repeated addition, multiplication in $\mathbb{N}$ could also be interpreted differently. Let's try $2 \times 3 = 6$.



$|A| = 2$ $\qquad$ $|B| = 3$

This set contains all the possible arrangements of the elements of A & B.

$|A \times B| = 6$ The size of this set is the result of the product of the sizes of the sets A & B.

Now for the other numbers the intuition of multiplication as repeated addition starts to loosen up a bit and is more seen as a mathematically correct map of 2 elements to another.

For example consider fractions ($\mathbb{Q}$):

$\frac{1}{2}$ of  $= \frac{1}{4}$ of a 

$\frac{1}{2}$ of a whole $\qquad$ Whole

$$\boxed{\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}}$$

Here, repeated addition can somehow be seen $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Consider the integers ($\mathbb{Z}$):

$-1 \times 2 = -2$, $2 \times 2 = 4$, $-2 \times -2 = +4$

$\cdot$	+	−
+	+	−
−	-	+

Def$\underline{\text{n}}$ for sign

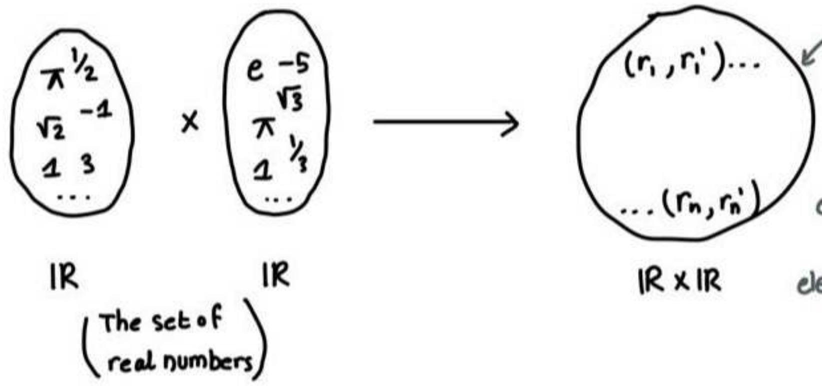
Now for $\mathbb{R}$ and $\mathbb{C}$, less and less repeated addition can be seen in their definitions. For other mathematical objects like vectors, matrices, sets, algebraic structures, the notion completely disappears.

e.g. Matrix Multiplication: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} ae + bf \\ ce + df \end{pmatrix}$

As the product is divorced from its rudimentary notion of repeated addition, we can define multiplication on other mathematical objects which are not numbers while remaining mathematically correct!

The product of sets previously encountered is known as the Cartesian Product.

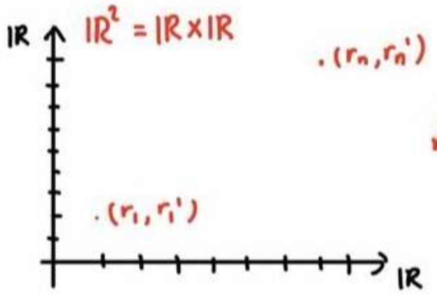
Consider $\mathbb{R} \times \mathbb{R}$:



This set contains all the possible unique combinations of the elements of $\mathbb{R}$.

Geometric Interpretation

We need a way to display all the possible combinations of $\mathbb{R}^2$. To achieve this, we could orient one real line perpendicular to the other so that each element of one line intersects the elements of the other exactly once.



The geometric interpretation of the Cartesian Product is the Cartesian Plane.

Let's try this with other sets:

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The following if done in the rigorous way would be out of the scope of this
   article. Instead, a sketch of the idea will be provided to capture the
essence of the problem. Full information will be present in the references.
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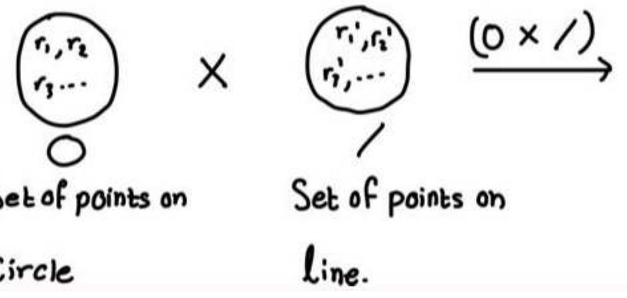
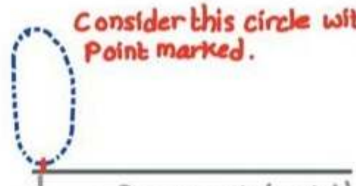
We shall consider a Circle and a line as a set of points satisfying their respective equations and that those solutions give rise to these particular shapes.



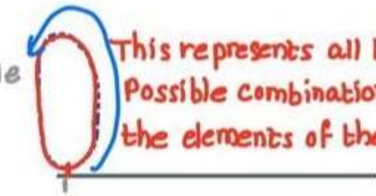
Similarly to the previous example, we place these two $\perp$ to each other and the space that this setup inscribes would give rise to a <u>Cylinder</u>. <u>Imagine it as follows</u>:



Consider this circle with one Point marked.

r_0 → Suppose it is at the beginning of the line at r_0.

We rotate the Circle $\xrightarrow{\text{360° on itself}}$ while leaving a red trail.



r_0

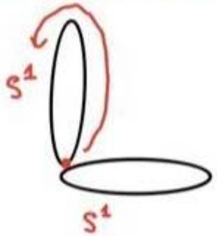
This represents all the Possible combinations of the elements of the circle and r_0.

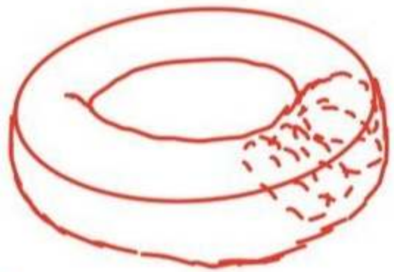
[This process is repeated an uncountably infinite no of times until a cylinder is generated.] → rotate through all r on the line.

This is the geometric representation of $(O \times /)$

$S^1 \times S^1$

Another interesting shape can be obtained when considering 2 circles $\perp$ to each other





$(O \times O)$ = Doughnut

Rotating the first circle throughout all the points of the other will end up generating something looking like a hollow doughnut.

This process can be applied to other shapes to generate new ones. The references contain other shapes which can be derived from the multiplication of simpler ones.

References:

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